SUMMERTIME SHELL BALANCES

WITH THE SMITH FAMILY

Written and Drawn
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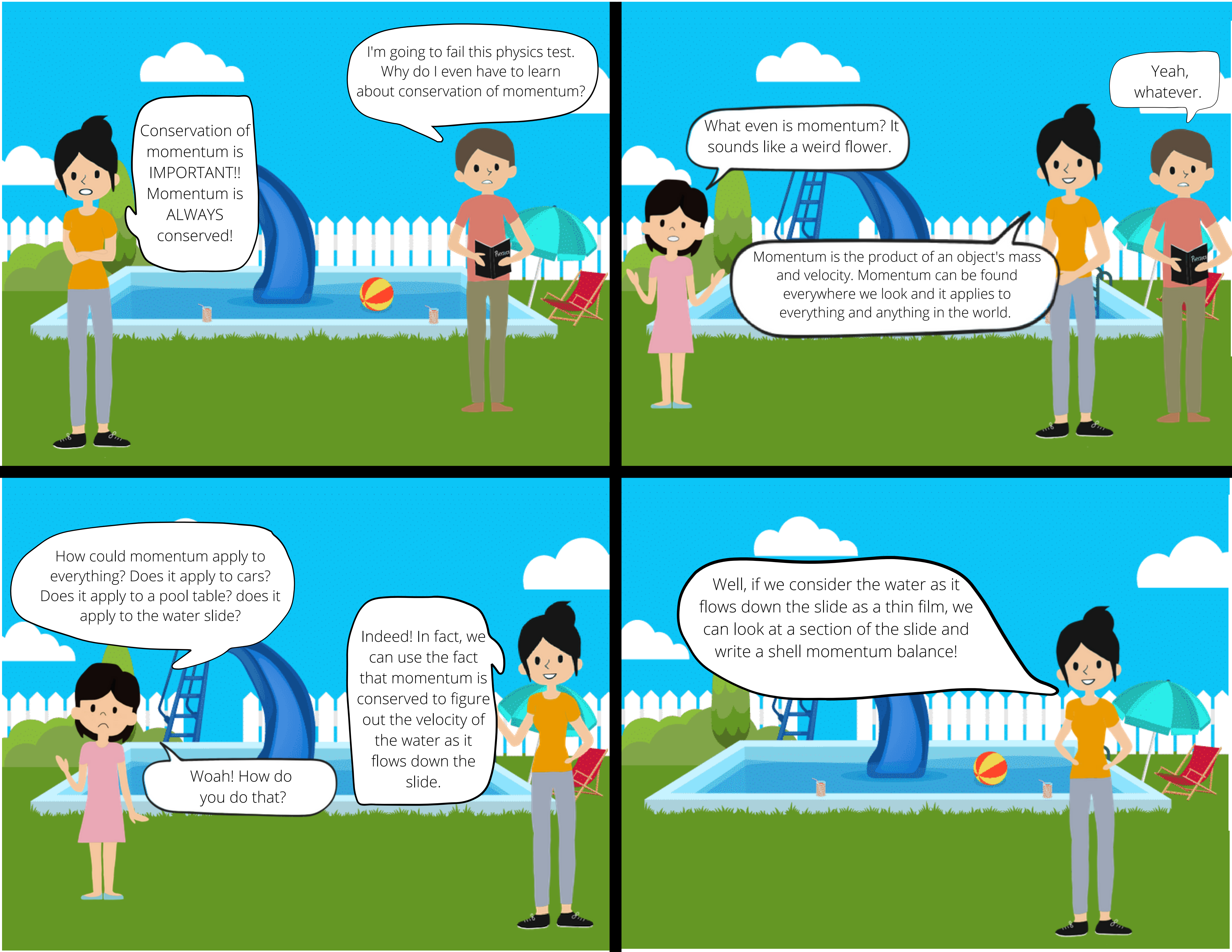
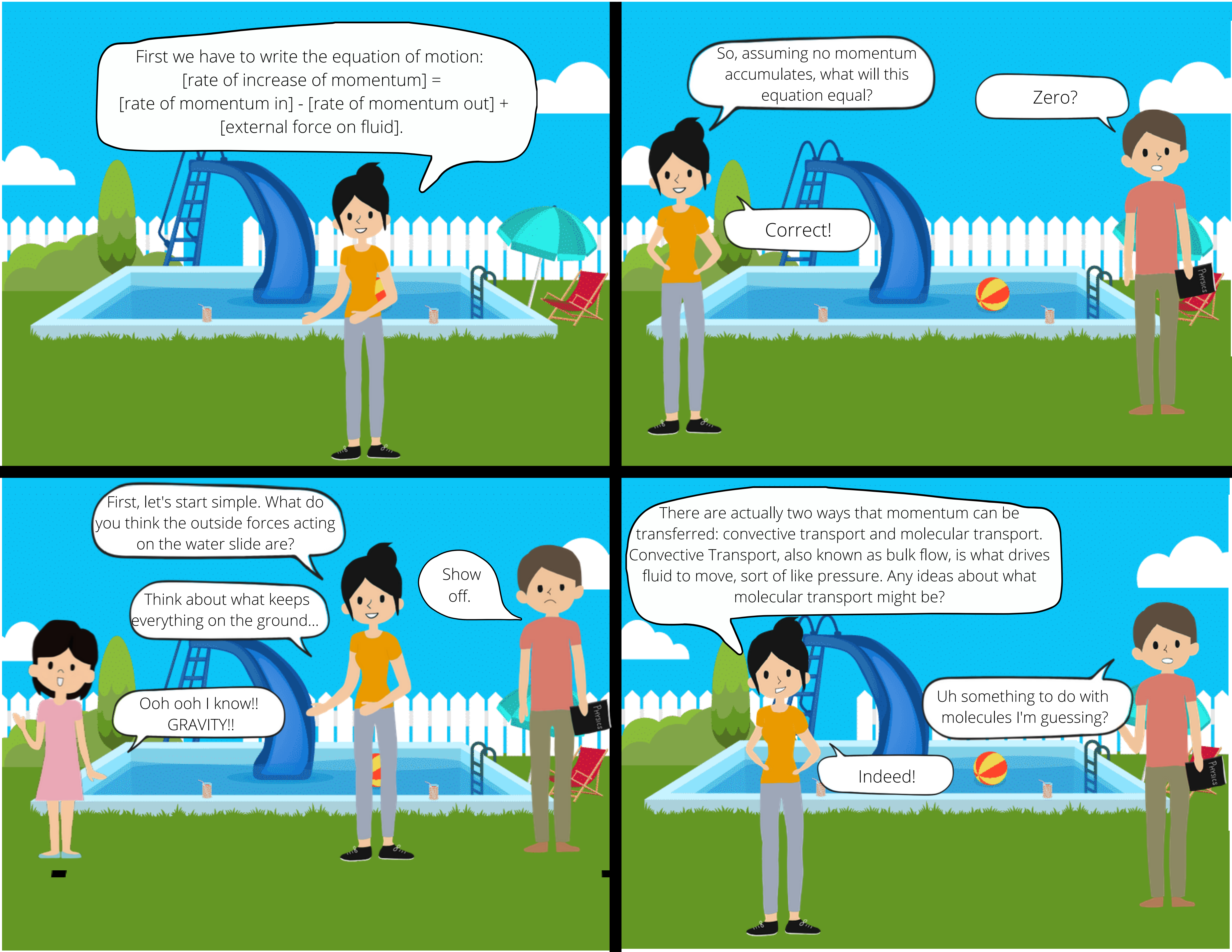
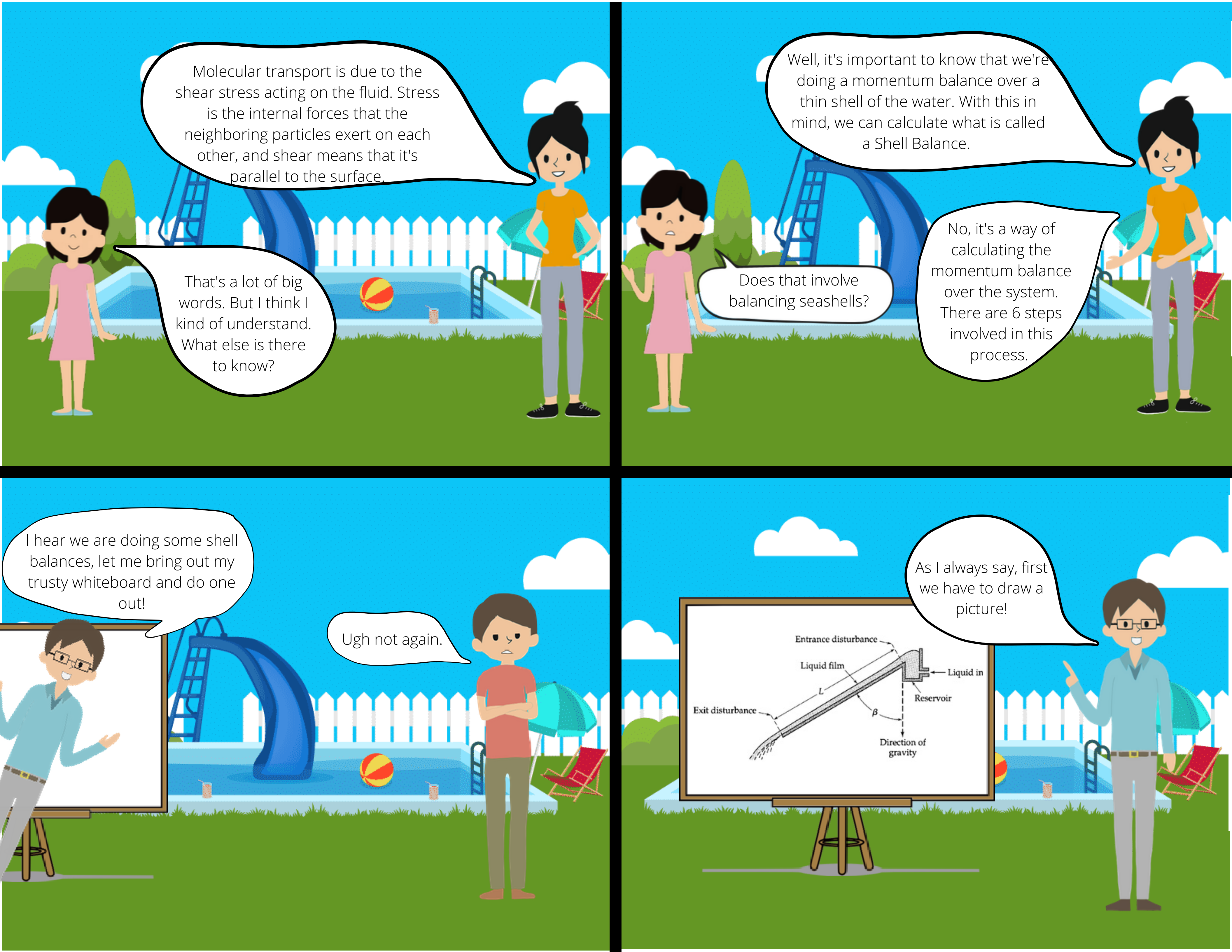
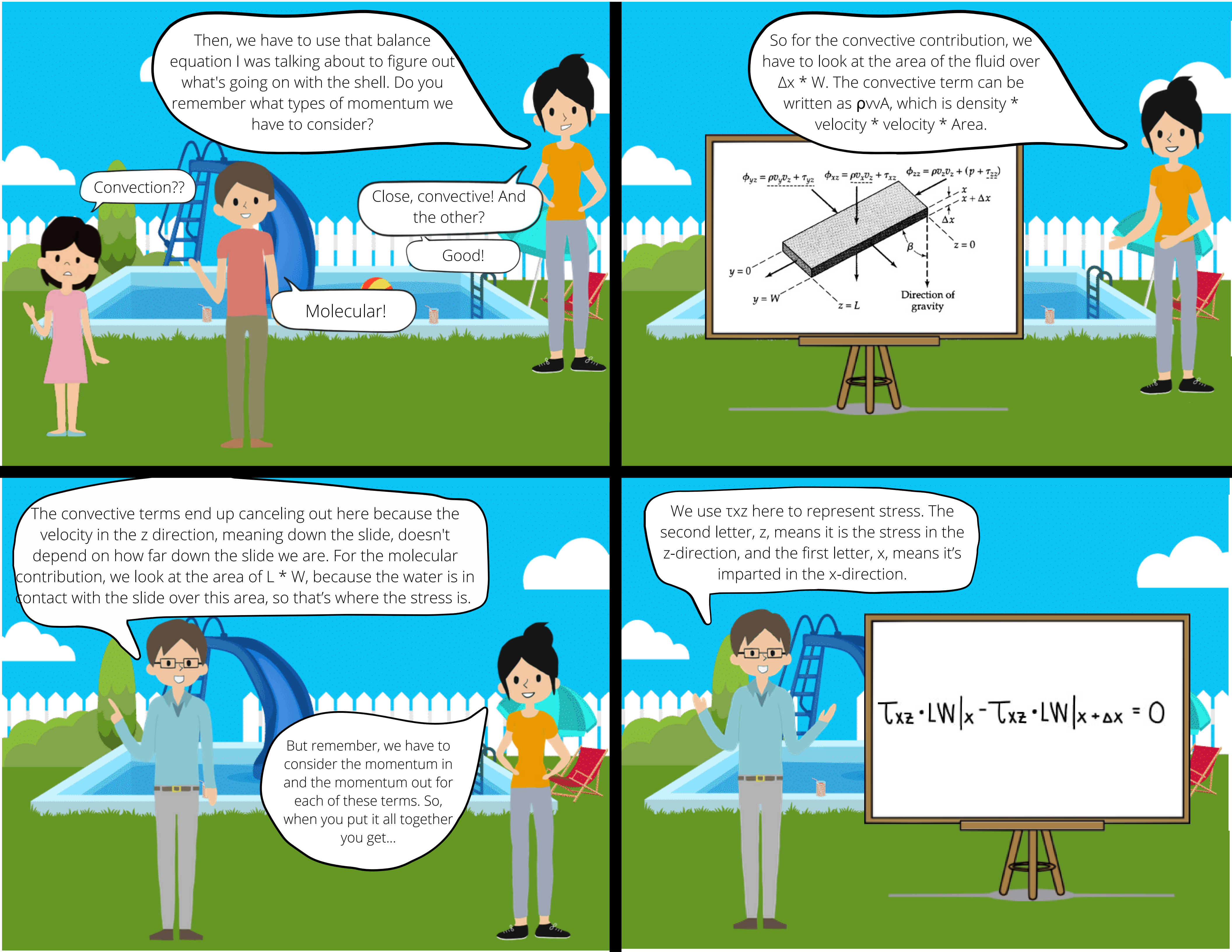
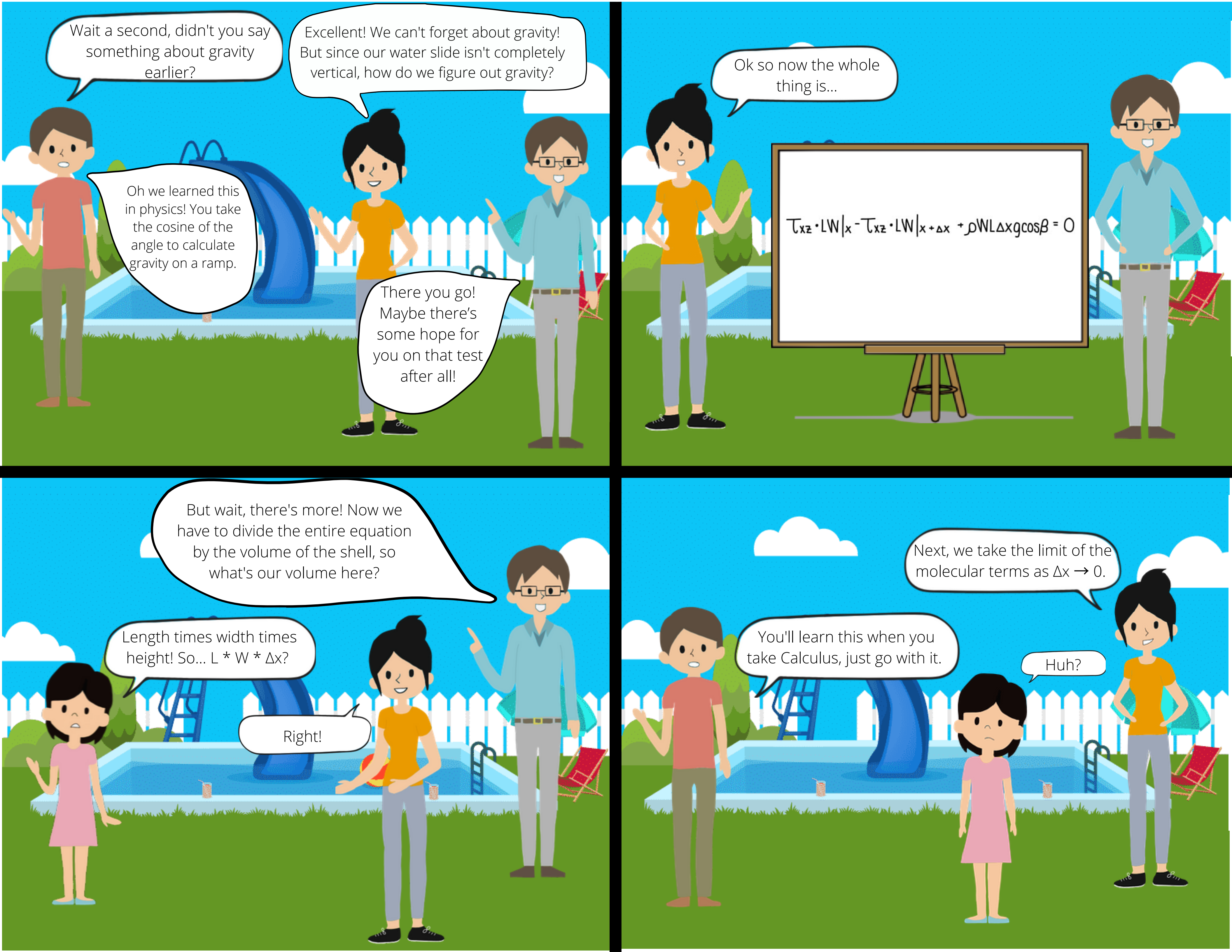
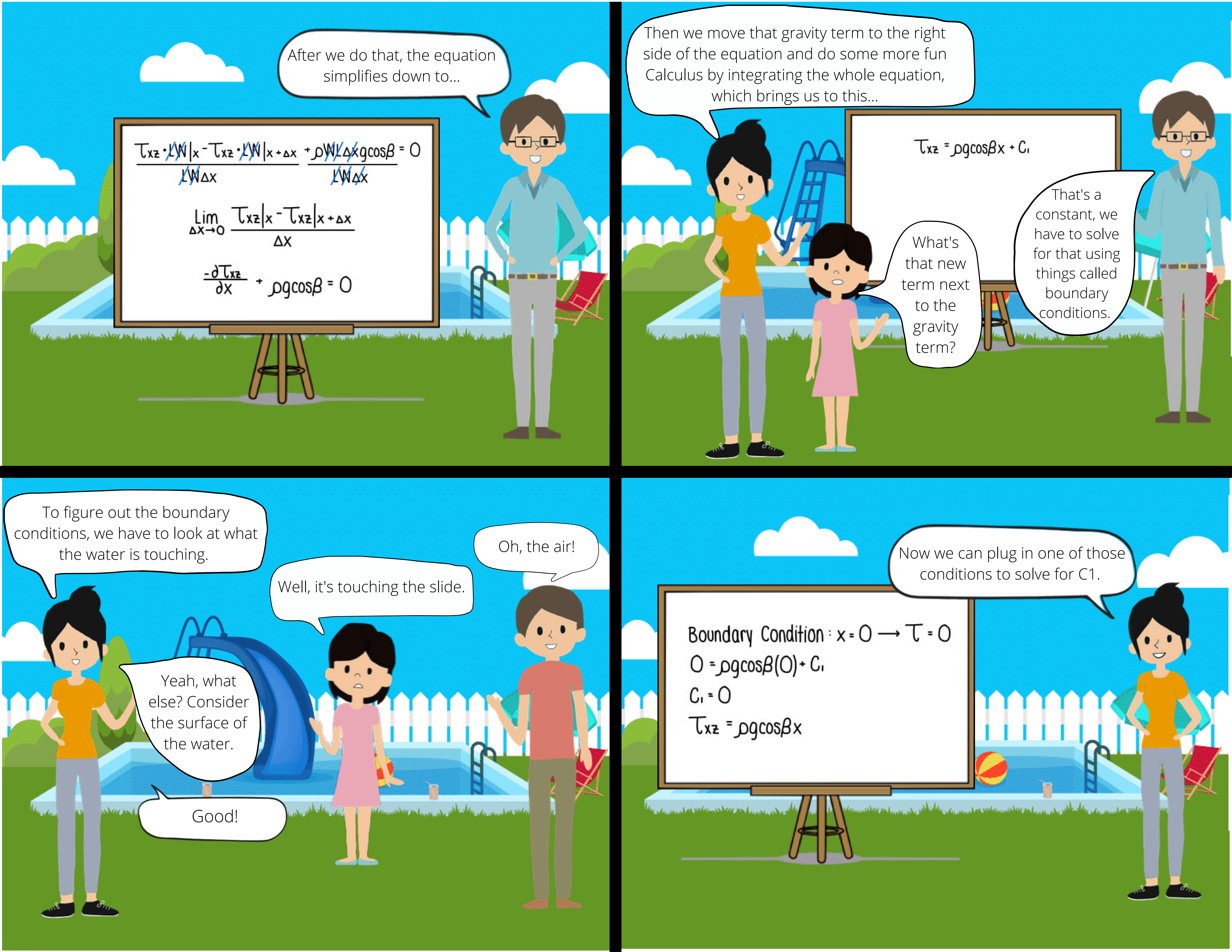

I'm going to fail this physics test. Why do I even have to learn about conservation of momentum?
Conservation of momentum is IMPORTANT!! Momentum is ALWAYS conserved!
What even is momentum? It sounds like a weird flower.
Yeah, whatever.
Momentum is the product of an object's mass and velocity. Momentum can be found everywhere we look and it applies to everything and anything in the world.
How could momentum apply to everything? Does it apply to cars? Does it apply to a pool table? does it apply to the water slide?
Indeed! In fact, we can use the fact that momentum is conserved to figure out the velocity of the water as it flows down the slide.
Woah! How do you do that?
Well, if we consider the water as it flows down the slide as a thin film, we can look at a section of the slide and write a shell momentum balance!


First we have to write the equation of motion: [rate of increase of momentum] = [rate of momentum in] - [rate of momentum out] + [external force on fluid].
So, assuming no momentum accumulates, what will this equation equal?
Zero?
Correct!
First, let's start simple. What do you think the outside forces acting on the water slide are?
Show off.
Think about what keeps everything on the ground...
Ooh ooh I know!! GRAVITY!!
There are actually two ways that momentum can be transferred: convective transport and molecular transport. Convective Transport, also known as bulk flow, is what drives fluid to move, sort of like pressure. Any ideas about what molecular transport might be?
Uh something to do with molecules I'm guessing?
Indeed!

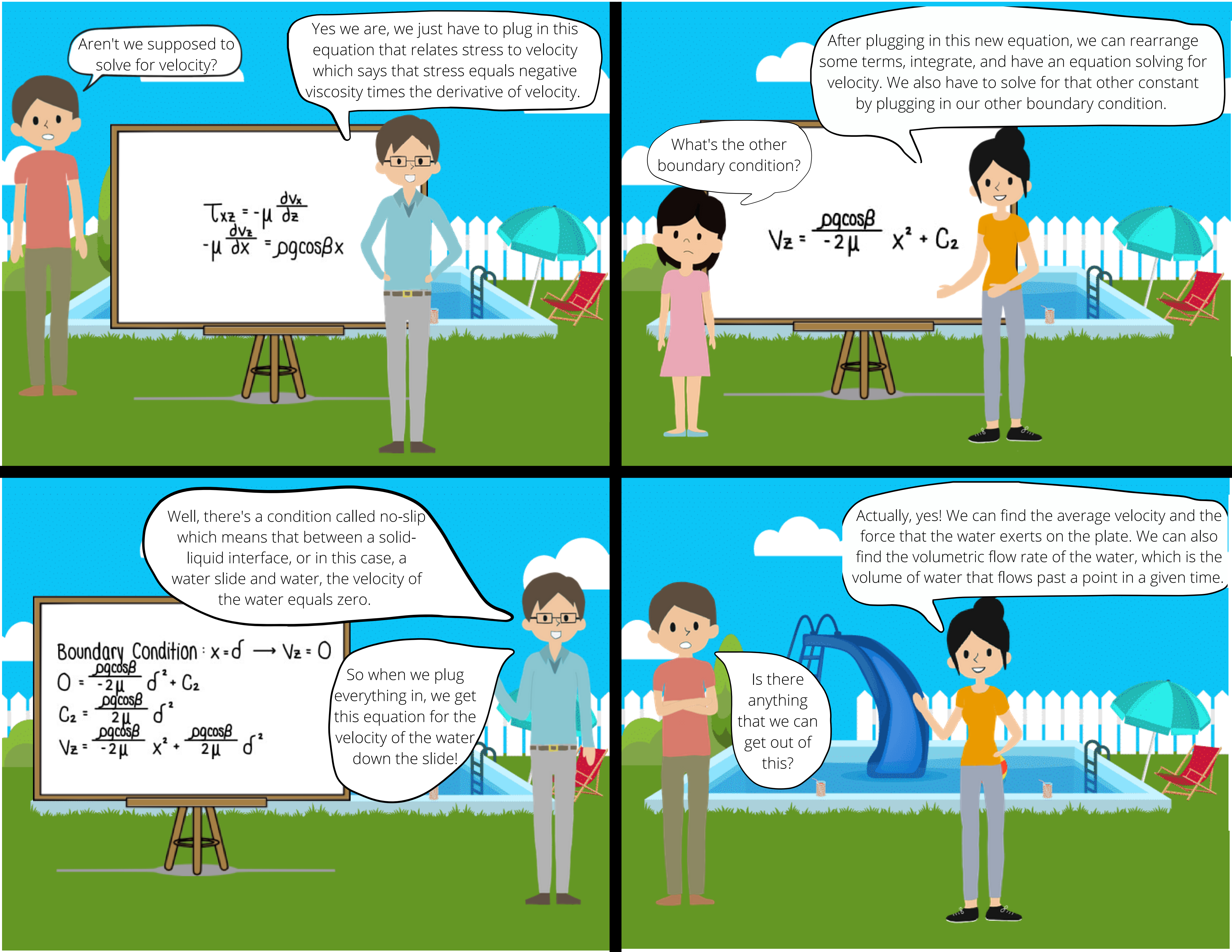

Molecular transport is due to the shear stress acting on the fluid. Stress is the internal forces that the neighboring particles exert on each other, and shear means that it's parallel to the surface.
That's a lot of big words. But I think I kind of understand. What else is there to know?
Well, it's important to know that we're doing a momentum balance over a thin shell of the water. With this in mind, we can calculate what is called a Shell Balance.
Does that involve balancing seashells?
No, it's a way of calculating the momentum balance over the system. There are 6 steps involved in this process.
I hear we are doing some shell balances, let me bring out my trusty whiteboard and do one out!
Ugh not again.
As I always say, first we have to draw a picture!
Entrance disturbance
Liquid film
Liquid in
Reservoir
Exit disturbance
L
β
Direction of gravity

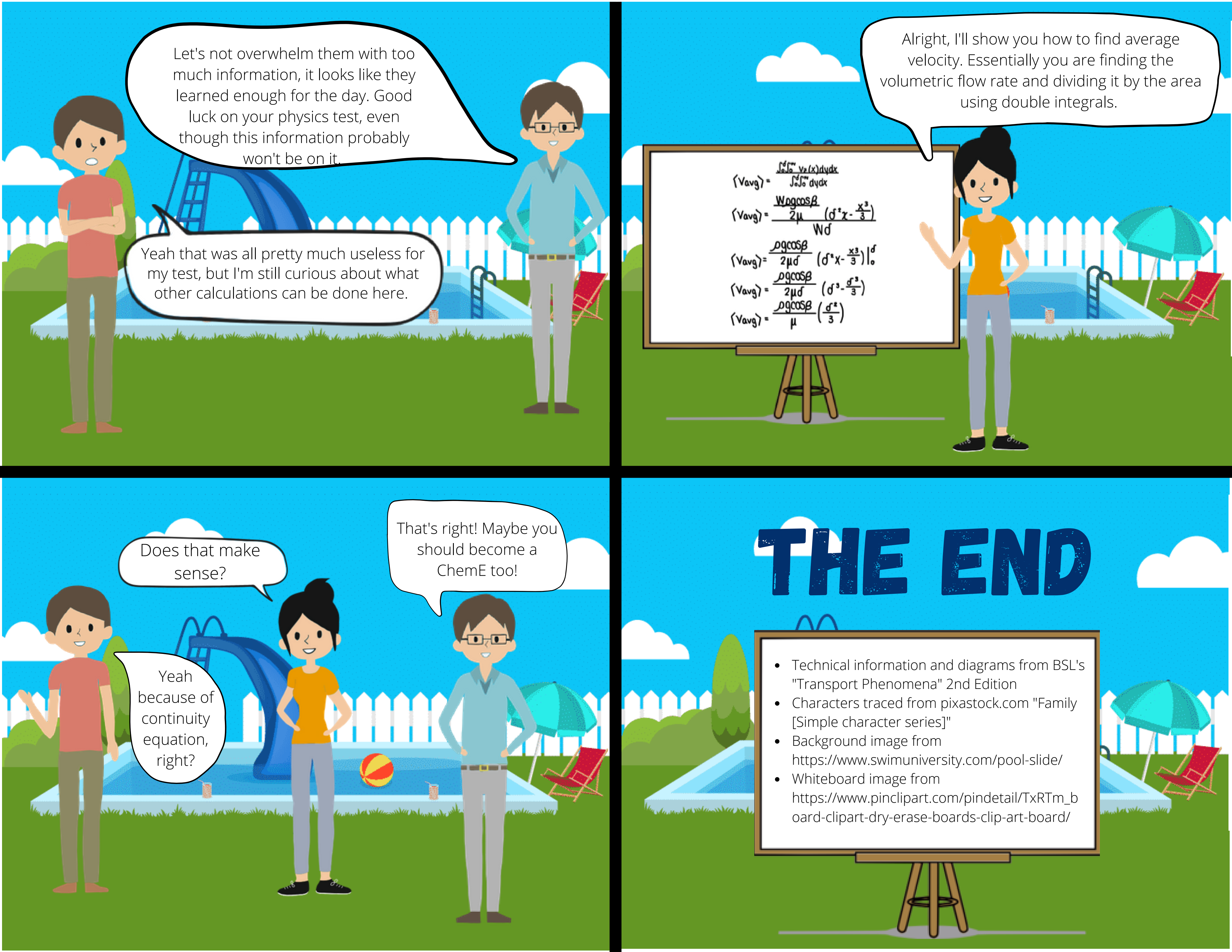

Then, we have to use that balance equation I was talking about to figure out what's going on with the shell. Do you remember what types of momentum we have to consider?
Convection??
Close, convective! And the other?
Molecular!
Good!
So for the convective contribution, we have to look at the area of the fluid over Δx * W. The convective term can be written as ρvvA, which is density * velocity * velocity * Area.
$\phi_{yz} = \rho v_y v_z + \tau_{yz}$
$\phi_{xz} = \rho v_x v_z + \tau_{xz}$
$\phi_{zz} = \rho v_z v_z + (p + \tau_{zz})$
x
$x + \Delta x$
Δx
$z = 0$
β
$y = 0$
$y = W$
$z = L$
Direction of gravity
The convective terms end up canceling out here because the velocity in the z direction, meaning down the slide, doesn't depend on how far down the slide we are. For the molecular contribution, we look at the area of L * W, because the water is in contact with the slide over this area, so that's where the stress is.
But remember, we have to consider the momentum in and the momentum out for each of these terms. So, when you put it all together you get...
We use τxz here to represent stress. The second letter, z, means it is the stress in the z-direction, and the first letter, x, means it's imparted in the x-direction.
$\tau_{xz} \cdot LW|_x - \tau_{xz} \cdot LW|_{x+\Delta x} = 0$


Wait a second, didn't you say something about gravity earlier?
Excellent! We can't forget about gravity! But since our water slide isn't completely vertical, how do we figure out gravity?
Oh we learned this in physics! You take the cosine of the angle to calculate gravity on a ramp.
There you go! Maybe there's some hope for you on that test after all!
Ok so now the whole thing is...
$\tau_{xz} \cdot LW|_x - \tau_{xz} \cdot LW|_{x+\Delta x} + \rho WL\Delta x g \cos\beta = 0$
But wait, there's more! Now we have to divide the entire equation by the volume of the shell, so what's our volume here?
Length times width times height! So... L * W * Δx?
Right!
Next, we take the limit of the molecular terms as Δx → 0.
You'll learn this when you take Calculus, just go with it.
Huh?



After we do that, the equation simplifies down to...
$\frac{\tau_{xz} \cdot \cancel{L}\cancel{W}|_x - \tau_{xz} \cdot \cancel{L}\cancel{W}|_{x+\Delta x}}{\cancel{L}\cancel{W}\Delta x} + \frac{\rho \cancel{W}\cancel{L}\cancel{\Delta x} g\cos\beta}{\cancel{L}\cancel{W}\cancel{\Delta x}} = 0$
$\lim_{\Delta x \to 0} \frac{\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}}{\Delta x}$
$\frac{-\partial \tau_{xz}}{\partial x} + \rho g\cos\beta = 0$
Then we move that gravity term to the right side of the equation and do some more fun Calculus by integrating the whole equation, which brings us to this...
$\tau_{xz} = \rho g\cos\beta x + C_1$
What's that new term next to the gravity term?
That's a constant, we have to solve for that using things called boundary conditions.
To figure out the boundary conditions, we have to look at what the water is touching.
Well, it's touching the slide.
Yeah, what else? Consider the surface of the water.
Oh, the air!
Good!
Now we can plug in one of those conditions to solve for C1.
Boundary Condition: $x = 0 \longrightarrow \tau = 0$
$0 = \rho g\cos\beta(0) + C_1$
$C_1 = 0$
$\tau_{xz} = \rho g\cos\beta x$


Aren't we supposed to solve for velocity?
Yes we are, we just have to plug in this equation that relates stress to velocity which says that stress equals negative viscosity times the derivative of velocity.
$\tau_{xz} = -\mu \frac{dv_x}{dz}$
$-\mu \frac{dv_z}{dx} = \rho g \cos\beta x$
After plugging in this new equation, we can rearrange some terms, integrate, and have an equation solving for velocity. We also have to solve for that other constant by plugging in our other boundary condition.
What's the other boundary condition?
$V_z = \frac{\rho g \cos\beta}{-2\mu} x^2 + C_2$
Well, there's a condition called no-slip which means that between a solid-liquid interface, or in this case, a water slide and water, the velocity of the water equals zero.
Boundary Condition : $x = \delta \longrightarrow V_z = 0$
$0 = \frac{\rho g \cos\beta}{-2\mu} \delta^2 + C_2$
$C_2 = \frac{\rho g \cos\beta}{2\mu} \delta^2$
$V_z = \frac{\rho g \cos\beta}{-2\mu} x^2 + \frac{\rho g \cos\beta}{2\mu} \delta^2$
So when we plug everything in, we get this equation for the velocity of the water down the slide!
Is there anything that we can get out of this?
Actually, yes! We can find the average velocity and the force that the water exerts on the plate. We can also find the volumetric flow rate of the water, which is the volume of water that flows past a point in a given time.



Let's not overwhelm them with too much information, it looks like they learned enough for the day. Good luck on your physics test, even though this information probably won't be on it.
Yeah that was all pretty much useless for my test, but I'm still curious about what other calculations can be done here.
Alright, I'll show you how to find average velocity. Essentially you are finding the volumetric flow rate and dividing it by the area using double integrals.
$\langle V_{avg} \rangle = \frac{\int_0^{\delta}\int_0^{W} v_z(x)\,dy\,dx}{\int_0^{\delta}\int_0^{W} dy\,dx}$
$\langle V_{avg} \rangle = \frac{\frac{W\rho g \cos\beta}{2\mu}\left(\delta^2 x - \frac{x^3}{3}\right)}{W\delta}$
$\langle V_{avg} \rangle = \frac{\rho g \cos\beta}{2\mu\delta}\left(\delta^2 x - \frac{x^3}{3}\right)\Big|_0^{\delta}$
$\langle V_{avg} \rangle = \frac{\rho g \cos\beta}{2\mu\delta}\left(\delta^3 - \frac{\delta^3}{3}\right)$
$\langle V_{avg} \rangle = \frac{\rho g \cos\beta}{\mu}\left(\frac{\delta^2}{3}\right)$
Does that make sense?
Yeah because of continuity equation, right?
That's right! Maybe you should become a ChemE too!
THE END
Technical information and diagrams from BSL's "Transport Phenomena" 2nd Edition
Characters traced from pixastock.com "Family [Simple character series]"
Background image from https://www.swimuniversity.com/pool-slide/
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