NOTIONS OF STABILITY OF SHEAVES

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1. Stability and Filtrations

1.1. Semistable sheaves. Let $X$ be a projective scheme over a field $k$ and $E$ be a coherent sheaf on $X$. The Euler characteristic of $E$ is denoted by $\chi(E) = \sum (-1)^i h^i(X,E)$, where $h^i(X,E) = \dim_k H^i(X,E)$. Fix $\mathcal{O}(1)$ as an ample line bundle on $X$.

**Definition and Lemma.** The Hilbert polynomial $P(E) : m \mapsto \chi(E \otimes \mathcal{O}(m))$ is a polynomial of $m$ and can be written as $P(E,m) = \sum \dim E_i \alpha_i(E)m^i$. 

**Note.** $\alpha_{\dim X}(\mathcal{O}_X)$ is exactly the degree of $X$ with respect to $\mathcal{O}(1)$. Furthermore, if $X$ is reduced and irreducible, of dimension $d_X$, then $\alpha_{d_X}(E) = \text{rank}(E) \cdot \alpha_{d_X}(\mathcal{O}_X)$.

**Definition 1.1.1.** The reduced Hilbert polynomial $p(E)$ of a coherent sheaf $E$ of dimension $d$ is defined by $p(E,m) = \frac{P(E,m)}{\alpha_d(E)}$.

For two polynomials $p(m)$ and $q(m)$, we say $p(m) < q(m)$ if that holds for $m >> 0$.

**Definition 1.1.2.** A coherent sheaf $E$ purely of dimension $d$ (i.e. every nonzero subsheaf is of support dimension $d$) is (semi)stable if for any proper subsheaf $F \subset E$, one has $p(F) < (\leq) p(E)$.

**Exercise 1.1.1.** $E$ is (semi)stable if and only if for all proper quotient sheaves $E \rightarrow G$ with $\alpha_d(G) > 0$, one has $p(E) < (\leq)p(G)$.

**Exercise 1.1.2.** Suppose $F, G$ are semistable, purely of dimension $d$. If $p(F) > p(G)$, then $\text{Hom}(F,G) = 0$; if $p(F) = p(G)$ and $f : F \rightarrow G$ is nontrivial, then $f$ is injective if $F$ is stable and surjective if $G$ is stable.

1.2. Slope stable. Let $X$ be a smooth projective curve over an algebraic closed field $k$ and $E$ be a locally free sheaf of rank $r$. Then $\chi(E) = \deg(E) + r(1 - g)$, where $g$ is the genus of $X$. So $P(E,m) = (\deg(X)m + \mu(E) + (1 - g)r)$, where $\mu(E) = \frac{\deg(E)}{r}$ is called the slope of $E$.

In this case, the stability means:

$E$ is (semi)stable if for all subsheaves $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$, one has $\mu(F) < (\leq)\mu(E)$.

In general, this becomes the $\mu$-stability. Denote $d = \dim X$.

**Definition 1.2.1.** Suppose that $E$ is a coherent sheaf of dimension $d = \dim X$. The degree of $E$ is defined to be $\deg(E) = \alpha_{d-1}(E) - \text{rank}(E) \cdot \alpha_{d-1}(\mathcal{O}_X)$.

And its slope is $\mu(E) = \frac{\deg(E)}{\text{rank}(E)}$.

**Definition 1.2.2.** A coherent sheaf $E$ of dimension $d = \dim(X)$ is $\mu$-(semi)stable if
(i) any torsion subsheaf of $E$ has support of codimension at least 2;
(ii) $\mu(F) < (\leq) \mu(E)$ for all subsheaves $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$.

**Exercise 1.2.1.**

- If $E$ is purely of dimension $d = \dim X$, then $\mu$-stable $\implies$ stable $\implies$ semistable $\implies$ $\mu$-semistable.
- Given $X$ being integral, if the coherent sheaf $E$ of dimension $d = \dim X$ is $\mu$-semistable, and $\text{rank}(E)$ is coprime to $\text{deg}(E)$, then $E$ is $\mu$-stable.

### 1.3. Harder-Narasimhan Filtration

**Definition 1.3.1.** Suppose a coherent sheaf $E$ over $X$ is purely of dimension $d$. A Harder-Narasimhan filtration for $E$ is an increasing filtration

$$0 = \text{HN}_0(E) \subset \text{HN}_1(E) \subset \cdots \subset \text{HN}_\ell(E) = E,$$

such that $\text{gr}^{\text{HN}}_i := \text{HN}_i(E)/\text{HN}_{i-1}(E)$ for $i = 1, \cdots, \ell$ are semistable sheaves of dimension $d$ with reduced Hilbert polynomials $p_i$ satisfying

$$p_{\text{max}}(E) : = p_1 > \cdots > p_\ell = : p_{\text{min}}(E).$$

**Theorem 1.3.1.** Every pure sheaf $E$ has a unique HN filtration.

**Proof.** We first need the following lemma.

**Lemma 1.3.1.** Suppose $E$ is purely of dimension $d$. Then there exists $F \subset E$ such that for all $G \subset E$, one has $p(F) \geq p(G)$, and in case of equality $F \supset G$. Moreover $F$ is unique and semistable. We call $F$ the maximal destabilizing sheaf of $E$.

**Proof of Lemma.** We define an order ‘$\leq$’ on the nontrivial subsheaves of $E$: $F_1 \leq F_2$ if $F_1 \subset F_2$ and $p(F_1) \leq p(F_2)$. We say a sheaf is $\leq$-maximal if it is maximal with respect to this order. By ascending property, for each $F \subset E$, there exists a subsheaf $F'$ such that $F \subset F' \subset E$ and $F'$ is $\leq$-maximal. Let $F \subset E$ be the $\leq$-maximal subsheaf with minimal $\alpha_d(F)$. We claim that $F$ has the asserted properties.

Suppose there exists $G \subset E$ with $p(G) \geq p(F)$. First we show that we can assume $G \subset F$ by replacing $G$ by $G \cap F$. Indeed, if $G \not\subset F$, $F$ is a proper subsheaf of $F + G$, so $p(F) > p(F + G)$. Consider

$$0 \to F \cap G \to F \oplus G \to F + G \to 0.$$

We have

$$P(F) + P(G) = P(F \cap G) + P(F + G),$$

$$\alpha_d(F) + \alpha_d(G) = \alpha_d(F \cap G) + \alpha_d(F + G).$$

Hence

$$\alpha_d(F \cap G)(p(G) - p(F \cap G)) = \alpha_d(F + G)(p(F + G) - p(F)) + (\alpha_d(G) - \alpha_d(F \cap G))(p(F) - p(G)).$$

Therefore $p(F) \leq p(G) < p(F \cap G)$.

Next, fix $G \subset F$ with $p(G) > p(F)$ which is $\leq$-maximal in $F$. Let $G'$ be the $\leq$-maximal sheaf in $E$ containing $G$. In particular, $p(F) < p(G) \leq p(G')$. By definition, $G' \not\subset F$ (otherwise $\alpha_d(G') < \alpha_d(F)$), hence $F$ is a proper subsheaf of $F + G'$. Therefore $p(F) > p(F + G')$. As before, we have $p(F \cap G') > p(G') \geq p(G)$. Since $G \subset F \cap G' \subset F$, this is a contradiction to the assumption on $G$.

The other two properties follow from the first property. □


Existence of HN-filtration: Let $E_1$ be the maximal destabilizing subsheaf. By induction, we can assume $E/E_1$ has an HN-filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{\ell-1} = E/E_1.$$ 

Let $E_{i+1} \subset E$ be the preimage of $G_i$. We just need to show $p(E_1) \geq p(E_2/E_1)$. This follows from the maximal property of $E_1$.

Uniqueness of HN-filtration: Assume $E$ and $E'$ are two HN-filtrations of $E$, with $p(E'_1) \geq p(E_1)$. Let $j$ be minimal number such that $E'_j \subset E_j$. Then

$$E'_j \rightarrow E_j \rightarrow E_j/E_{j-1}$$

in a nontrivial morphism between two semistable sheaves. Hence

$$p(E_j/E_{j-1}) \geq p(E'_j) \geq p(E_1) \geq p(E_j/E_{j-1}).$$

So $j = 1$ and $E'_1 \subset E_1$. Then $p(E'_1) \leq p(E_1)$. Repeat the argument, we can see $E'_1 = E_1$. Now by induction, $E/E_1$ has a unique HN-filtration. □

1.4. Jordan-Holder Filtration.

Definition 1.4.1. Let $E$ be a semistable coherent sheaf of dimension $d$ on $X$. A Jordan-Holder filtration is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

such that $\text{gr}_i(E) = E_i/E_{i-1}$ are stable with reduced Hilbert polynomial $p(E)$.

Proposition 1.4.1. JH-filtration exists and $\text{gr } E := \bigoplus_i \text{gr}_i(E)$ is independent of the choice of the JH-filtration.

Proof. The existence is straightforward: any filtration of $E$ by semistable sheaves with reduced Hilbert polynomial $p(E)$ has a maximal refinement, whose factors are necessarily stable.

The second statement follows from the same idea as in the proof of the uniqueness of the HN-filtration. We refer to Section 1.5 of Huybrechts and Lehn’s book for detail. □

Definition 1.4.2. Two semistable sheaves $E_1$ and $E_2$ with $p(E_1) = p(E_2)$ are $S$-equivalent if $\text{gr}(E_1) \cong \text{gr}(E_2)$.

Definition 1.4.3. A semistable sheaf $E$ is called polystable if $E$ is the direct sum of stable sheaves.

1.5. Relative case.

Theorem 1.5.1. Let $S$ be an integral $k$-scheme of finite type, $f : X \rightarrow S$ a projective morphism, $O_X(1)$ an $f$-ample invertible sheaf on $X$, and $F$ a flat family of $d$-dimensional coherent sheaves on the fibers of $f$. Then there is a projective birational morphism $g : T \rightarrow S$ of integral $k$-schemes and a filtration

$$0 = HN_0(T)(F) \subset HN_1(T)(F) \subset \cdots \subset HN_\ell(T)(F) = F_T,$$

such that

(i) $HN_i(T)(F)/HN_{i-1}(T)(F)$ are $T$-flat for all $i = 1, \cdots, \ell$;

(ii) there is a dense open subscheme $U \subset T$ such that $HN_{i,T}(F)_t = g^*_X(HN_{i,T}(F_{g(t)})$ for all $t \in U$.

Moreover, $(g, HN_{i,T}(F))$ is universal, meaning that if $g' : T' \rightarrow S$ is any dominant morphism of integral schemes, and $F'$ is a filtration of $F_{g'}$ satisfying the above two properties, then there exists an $S$-morphism $h : T' \rightarrow T$ with $F' = h^*_X(HN_{i,T}(F)).$
Sketch of proof. Just like the proof of the existence of the HN-filtration, the idea is to construct a family of sheaves which is generically the maximal destabilizing sheaf fiberwise. The main ingredient is the quot schemes. We refer to Section 2.3 of Huybrechts and Lehn’s book for detail.

\[ \square \]

Note. 1) In the proof, it can be shown that there exists a subscheme \( V \) of certain quot scheme \( \text{Quot} \) such that \( U \) is isomorphic to an open dense subscheme of \( S \), and \( T \) is taken to be closure of \( V \) in \( \text{Quot} \). So a priori, \( T \) is only birational to \( S \). It is interesting to try to find an example in which this is necessarily birational.

2) In condition ii), we can’t always take \( U = T \), since the graded quotients of the relative HN-filtration may degenerate to unstable sheaves on special fibers.

2. Examples of stable vector bundles

2.1. \( \Omega_{\mathbb{P}^n} \).

Proposition 2.1.1. \( \Omega_{\mathbb{P}^n} \) is stable.

Proof. By the uniqueness of HN-filtration, it is invariant under the \( SL(V) \)-action on \( \mathbb{P}^n = \mathbb{P}(V) \). In particular, every subsheaf in the filtration is a subbundle. However, since \( SL(V) \) acts transitively on \( \mathbb{P}^n \), and the induced action on the cotangent vectors at a fixed point is irreducible, the only nontrivial invariant subbundle is \( \Omega_{\mathbb{P}^n} \). Hence the HN-filtration is trivial and \( \Omega_{\mathbb{P}^n} \) is semistable. Now \( \gcd(\text{rank} \, \Omega_{\mathbb{P}^n}, \deg \, \Omega_{\mathbb{P}^n}) = 1 \), so it is \( \mu \)-stable, and hence stable. \( \square \)

2.2. \( \mathbb{P}^1 \times \mathbb{P}^1 \) and change of polarization. On \( \mathbb{P}^1 \times \mathbb{P}^1 \), it is easy to compute that

\[ \text{Ext}^1(\mathcal{O}(0,3), \mathcal{O}(1,-3)) \cong k^{10}. \]

So we can consider the sheaf \( E \) given by a non-trivial extension

\[ 0 \rightarrow \mathcal{O}(1,-3) \rightarrow E \rightarrow \mathcal{O}(0,3) \rightarrow 0. \]

Note that \( c_1(E) = (1,0) \), \( c_2(E) = 3 \). Let \( L = \mathcal{O}(1,5), L' = \mathcal{O}(1,7) \). We claim:

Proposition 2.2.1. (i) \( E \) is not \( L' \)-semistable.

(ii) \( E \) is \( L \)-stable.

Proof. (i) \( \mu_{L'}(\mathcal{O}(1,-3)) = 4 > \mu_{L'}(E) = \frac{7}{2} \).

(ii) We need to show that for any rank 1 subbundle \( \mathcal{O}(D) \) of \( E \), we have \( D \cdot L < \frac{5}{2} = \mu_L(E) \).

There are two cases:

(a) \( \mathcal{O}(D) \rightarrow \mathcal{O}(1,-3) \), or

(b) \( \mathcal{O}(D) \rightarrow \mathcal{O}(0,3) \).

For case (a), \( D \cdot L \leq \mathcal{O}(1,-3) \cdot \mathcal{O}(1,5) = 2 \).

For case (b), let \( D = (\alpha, \beta) \), then \( \alpha \leq 0 \) and \( \beta \leq 3 \). \((\alpha, \beta) \neq (0,3) \) since the extension is nontrivial. Hence \( D \cdot L = 5\alpha + \beta \leq 2 \). \( \square \)