KAZHHDAN-LUSZTIG CELLS

SIDDHARTH VENKATESH

ABSTRACT. These are notes for a talk on Kazhdan-Lusztig Cells for Hecke Algebras. In this talk, we construct the Kazhdan-Lusztig basis for the Hecke algebra associated to an arbitrary Coxeter group, in full multiparameter generality. We then use this basis to construct a partition of the Coxeter group into the Kazhdan-Lusztig cells and describe the corresponding cell representations. Finally, we specialize the construction to the case of the symmetric group. The main references for the talk are [Lus14, GJ11, Wil].

CONTENTS

1. Hecke Algebra associated to a Weighted Coxeter Group 1
2. The Bar Involution and the Kazhdan-Lusztig Basis 2
3. Cells and Cell Representations 4
4. Examples of Cells: The case of Type A 5
References 6

1. Hecke Algebra associated to a Weighted Coxeter Group

We begin by defining the notion of a weighted Coxeter group.

Definition 1.1. Let $W,S$ be a Coxeter system. Let $l : W \to \mathbb{Z}$ be the length function of the Coxeter group. Then, a weight function on $W$ is a map $L : W \to \mathbb{Z}$ such that

$$l(ww') = l(w) + l(w') \Rightarrow L(ww') = L(w) + L(w').$$

We call the pair $(W,L)$ a weighted Coxeter group.

Remark. Note that the additivity condition on the weight function is equivalent to the statement that a weight function is additive on reduced decompositions in $W$ and is hence determined by its values on $S$. In fact, a weight function can be specified by giving arbitrary weights to elements in $S$ subject to the sole condition that if $m_{st}$ is odd, then $L(s) = L(t)$.

Remark. Because reduced decompositions for $w^{-1}$ are obtained by reversing reduced decompositions for $w$, we have $L(w) = L(w^{-1})$.

Throughout the rest of the talk, let us fix a weighted Coxeter group $W,L$. Fix some field $k$ of characteristic 0. We now define the generic Iwahori-Hecke algebra associated to $W,L$.

Definition 1.2. Let $A = k[q,q^{-1}]$ be the algebra of Laurent polynomials over $k$ and for $s \in S$, let $q_s = q^{L(s)}$. Then, the (generic) Iwahori-Hecke algebra $H$ associated to $W$, is the $A$-algebra with generators $\{T_s : s \in S\}$ and relations

1. Eigenvalue Relation: $(T_s - q_s)(T_s + q_s^{-1}) = 0$
2. Braid Relation: $T_sT_t \cdots = T_tT_s \cdots$ (with $m_{st}$ many factors on each side).

As a consequence of the defining relations we have
Proposition 1.3. \( \mathcal{H} \) is free over \( A \) with basis \( T_w \). In this basis, the multiplication formula can be described as follows. For \( s \in S, w \in W \)

\[
T_sT_w = \begin{cases} 
T_{sw} & \text{if } l(sw) > l(w) \\
T_{sw} + (q_s - q_s^{-1})T_w & \text{if } l(sw) < l(w) 
\end{cases}
\]

2. The Bar Involution and the Kazhdan-Lusztig Basis

Let \( a \mapsto \bar{a} : A \to A \) be the \( k \)-algebra involution defined by sending \( q \) to \( q^{-1} \). \( a \mapsto \bar{a} \) extends to a semilinear involution on \( \mathcal{H} \) as follows:

Proposition 2.1. There is a unique \((A, \bar{\cdot})\)-semilinear ring homomorphism \( x \mapsto \bar{x} : \mathcal{H} \to \mathcal{H} \) defined by sending \( T_s \mapsto T_s^{-1} \). This homomorphism is involutive and sends \( T_w \) to \( T_w^{-1} \) for each \( w \in W \).

This map is known as the bar involution on \( \mathcal{H} \).

Definition 2.2. For \( w, y \in W \) we define \( r_{w,y} \in A \) by

\[
\bar{T}_w = \sum_{y \in W} r_{y,w}T_y.
\]

Remark. Note that \( r_{w,w} = 1 \).

Using the bar involution, we can now construct the Kazhdan-Lusztig basis for \( \mathcal{H} \).

Definition 2.3. For an integer \( n \), define

\[
A_{\leq n} = \bigoplus_{m \leq n} kq^m.
\]

Similarly define \( A_{\geq n}, A_{< n}, A_{> n} \). With this definition in hand, define

\[
\mathcal{H}_{\leq 0} = \bigoplus_w A_{\leq 0}T_w
\]

and

\[
\mathcal{H}_{< 0} = \bigoplus_w A_{< 0}T_w.
\]

Theorem 2.4. (Kazhdan-Lusztig Basis) Let \( w \in W \). There exists a unique element \( C_w \in \mathcal{H}_{\leq 0} \) such that

\[
\bar{C}_w = C_w \text{ and } C_w \equiv T_w \mod \mathcal{H}_{< 0}.
\]

Additionally, \( C_w \in T_w + \sum_{y < w} A_{< 0}T_y \) (where \( y < w \) is in the Bruhat-Chevalley order on \( W \)) and \( \{ C_w : w \in W \} \) is an \( A \)-basis for \( \mathcal{H} \).

Proof. To prove the theorem, we need to prove the following Lemma regarding \( r_{w,y} \). Before stating the lemma, recall the Bruhat-Chevalley order on \( W \): \( x \leq y \) if \( x \) can be obtained from a reduced expression for \( y \) by removing some of the elements of \( S \). Note that \( x \leq y \) implies that \( l(x) \leq l(y) \) with equality if and only if \( x = y \). Now,

Lemma 2.5. The following two properties hold:
1. For any $x, z \in W$, 
\[ \sum_{y \in W} \bar{r}_{x,y} r_{y,z} = \delta_{x,z}. \]

2. For any $x, y \in W$, let $s \in S$ be such that $y > s y$. Then, 
\[ r_{x,y} = \begin{cases} r_{sx,sy} & \text{if } sx < x \\ r_{sx,sy} + (v_s - v_x^{-1})r_{x,sy} & \text{if } sy > y \end{cases}. \]

3. If $r_{x,y} \neq 0$, then $x \leq y$.

Proof of Lemma. Property 1 follows from the fact that $\bar{\cdot}$ is an involution. Property 2 follows from the formula for $T_s T_w$ using the fact that $\bar{\cdot}$ is multiplicative. To prove property 3, we induct on the length of $y$. The case of $l(y) = 0$ is obvious. So suppose $l(y) > 0$. Choose some $s$ such that $sx < x$. Suppose first that $sx < x$. Then, by property 2, 
\[ r_{sx,sy} = r_{x,y} \neq 0 \quad \text{and hence by induction } sx \leq sy \quad \text{which implies that } x \leq y. \]

On the other hand, if $sx > x$, then by property 2, either $r_{sx,sy} \neq 0$ or $r_{x,sy} \neq 0$. In the first case, by induction, $x \leq sx \leq sy < y$ and in the second case $x \leq sy < y$. This proves the Lemma.

We now return to the proof of the existence and uniqueness of the Kazhdan-Lusztig basis. We first prove existence. Fix $w \in W$. For any $x \leq w$, we construct an element $u_x \in A_{\leq 0}$ such that
1. $u_w = 1$.
2. for $x < w$, $u_x \in A_{< 0}$ and
\[ \bar{u}_x - u_x = \sum_{y : x < y \leq w} r_{x,y} u_y. \]

We induct on $l(w) - l(x) \geq 0$. For 0, $x = w$ and hence $u_x = u_w$. By the inductive hypothesis, $u_y$ is defined for all $y \leq w$ such that $l(y) > l(x)$ and satisfies the above properties. Hence, the term 
\[ a_x = \sum_{y : x < y \leq w} r_{x,y} u_y \]

is defined. We show that $a_x + \bar{a}_x = 0$. This follows from the previous Lemma and the following computation:
\[
\begin{align*}
    a_x + \bar{a}_x & = \sum_{y : x < y \leq w} r_{x,y} u_y + \bar{r}_{x,y} (u_y + \sum_{z : y < z \leq w} r_{y,z} u_z) \\
    & = \sum_{z : x < z \leq w} r_{x,z} u_z + \sum_{z : x < z \leq w} \bar{r}_{x,z} u_z + \sum_{z : x < z \leq w} \sum_{y : x < y < z} \bar{r}_{x,y} r_{y,z} u_z \\
    & = \sum_{z : x < z \leq w} r_{x,z} u_z + \sum_{z : x < z \leq w} \bar{r}_{x,z} u_z + \sum_{z : x < z \leq w} \delta_{x,z} u_z - r_{x,z} u_z + \bar{r}_{x,z} u_z = 0
\end{align*}
\]

Hence, $a_x = \sum_{n \in \mathbb{Z}} c_n q^n$ where $c_n + c_{-n} = 0$. Define 
\[ u_x := -\sum_{n < 0} c_n q^n. \]

Then, $u_x$ satisfies properties 1 and 2, as desired. Now, define the Kazhdan-Lusztig element associated to $w$ as
Clearly, $C_w$ satisfies the properties stated in the theorem, apart perhaps from invariance under the bar involution. This we verify with the following calculation:

\[
\tilde{C}_w = \sum_{y : y \leq w} \bar{u}_y T_y = \sum_{y : y \leq w} \bar{u}_y \sum_{x : x \leq y} \bar{r}_{x,y} T_x = \sum_{x : x \leq w} \left( \sum_{y : x \leq y \leq w} \bar{r}_{x,y} \bar{u}_y \right) T_x
\]

This completes the proof of existence. To prove uniqueness, it suffices to prove that if $h \in \mathcal{H}_{<0}$ satisfies $\bar{h} = h$, then $h = 0$. Since $h \in \mathcal{H}_{<0}$, we can write $h$ uniquely as $\sum_{y \in W} f_y T_y$, where $f_y \in A_{<0}$. Suppose for contradiction that not all $f_y = 0$. Choose $y_0$ with $f_{y_0} \neq 0$ maximal among such in the Bruhat-Chevalley order. Then, since $h$ is bar invariant, we have

\[
\sum_{y} f_y T_y = \sum_{y} \bar{f}_y \bar{r}_{x,y} T_x.
\]

Since $r_{y_0,y_0} = 1$ and $r_{y,y_0} = 0$ for all $y < y_0$, we see that the coefficient of $T_{y_0}$ on the left is $f_y$ and on the right is $\bar{f}_y$, which are not equal. This gives us a contradiction. Hence, $h = 0$ and we have uniqueness.

The last statement of the theorem is obvious. By construction and uniqueness, $C_w$ has the desired form and by upper triangularity (with respect to the Bruhat-Chevalley order), \{ $C_w : w \in W$ \} is a basis for $\mathcal{H}$ over $A$.

\[\square\]

3. Cells and Cell Representations

The Kazhdan-Lusztig basis of a Hecke algebra can be computed recursively but is difficult to compute. However, we can now use this basis to construct cells on the Coxeter group which has a much nicer description. We begin with an abstract definition of cells.

**Definition 3.1.** Let $A$ be an associative algebra with a basis \{ $a_w : w \in W$ \} indexed by a weighted Coxeter group $W, L$. We say that an ideal in $A$ is based if it is spanned by basis elements $a_w$. For, $x \in W$ we define three ideals $I_{x,L}, I_{x,R}, I_{x,LR}$ which are respectively the left, right and two-sided based ideals generated by $a_x$.

Define the preorder $\leq_L$ (resp. $\leq_R$, resp. $\leq_{LR}$) as $x \leq_L y$ if $a_x \in I_{y,L}$ (resp. $a_x \in I_{y,R}$, resp. $a_x \in I_{y,LR}$.) Let $\sim_L$, (resp. $\sim_R$, resp. $\sim_{LR}$) be the corresponding equivalence relations. Then, we call the corresponding equivalence classes the left cells (resp. right cells, resp. two-sided cells) of $W$ (with respect to $A$ and its chosen basis).

**Remark.** Note that $x \sim_L y$ if and only if they generate the same based left ideal (and similarly for the other two relations).

We now apply this definition to $A = \mathcal{H}$. If we use the standard basis, however, we only get one left, right or two sided cell (because the basis elements $T_w$ are all invertible). Instead, we apply the definition to the Kazhdan-Lusztig basis of $\mathcal{H}$. The resulting cells are called the (left, right, two-sided) Kazhdan-Lusztig cells, which we will abbreviate as KL cells. In the case of $\mathcal{H}$, we will also use $\mathcal{H}_{\leq_L}$ to denote $I_{x,L}$ and similarly for the right and two-sided ideals.
Remark. The map $w \mapsto w^{-1}$ carries left cells to right cells and vice versa. This is because the map $C_w \mapsto C_{w^{-1}}$ defines an anti-involution on $\mathcal{H}$.

From now on, the preorders and cells are defined with respect to the Kazhdan-Lusztig basis. We now use cells to construct representations of $\mathcal{H}$. We begin by introducing some notation:

Definition 3.2. Let $w \in W$. Define

$$\mathcal{H}_{\leq Lw} = \bigoplus_{x \leq w} AC_x$$

and define similar notions for the right and two-sided relations.

Note that all of the above constructions depend only on the cell of $w$ and hence we also use the notation $\mathcal{H}_{\leq Lw}$ where $C$ is the cell corresponding to $w$. Additionally, both $\mathcal{H}_{\leq Lw}$ and $\mathcal{H}_{< Lw}$ are left ideals in $\mathcal{H}$. Hence, we have the following definition:

Definition 3.3. Define the left cell module associated to $C$ as $L_C = \mathcal{H}_{\leq Lw}/\mathcal{H}_{< Lw}$. Similarly, define $R_C$ and $LR_C$. Note that the above proposition shows that these are respectively left, right and two-sided $\mathcal{H}$-modules.

Finally, note that the definition of cells immediately implies the following decomposition.

Proposition 3.4. As a left $\mathcal{H}$-module (after base changing to $k(q)$), we have

$$\mathcal{H} \cong \bigoplus_{C} L_C.$$  

We have similar decompositions over $R_C$ and $LR_C$.

Proof. This follows from the fact that $L_C$ has $\{C_w : w \in C\}$ as a basis and that $W = \sqcup C$ with disjoint union taken over all cells. \hfill $\Box$

4. Examples of Cells: The case of Type A

We end the talk by describing the left, right and two-sided cells (and the corresponding modules) in Type A i.e. when $W = S_n$ for some $n$. Before giving this description, we have to recall the RSK algorithm.

Definition 4.1. (Row Bumping Algorithm) Let $T$ be a semistandard Young tableau and let $i$ be a positive integer. We describe a new semistandard Young tableau denoted $T \leftarrow i$ as follows:

If $i$ is greater than or equal to every element in row 1, then $i$ is added in a new box at the end of row 1. Otherwise, $i$ replaces the leftmost number greater than $i$. This new number, $i_2$, is then added to row 2 in the same manner. The process continues until one of the numbers is added at the end of a row (which may have been of length 0 in $T$)

This algorithm is called the Row Bumping Algorithm.

Definition 4.2. (RSK Correspondence) Let $w \in S_n$ and let the one-line notation of $w$ be $w_1 \cdots w_n$, where $w_i = w(i)$. The RSK algorithm inductively defines a pair of standard Young tableau, $P_i, Q_i$ as follows:

1. $P_0 = Q_0 = \phi$.
2. $P_{i+1} = P_i \leftarrow w_{i+1}$.
3. $Q_{i+1}$ adds a box labelled with $i + 1$ in the location of $P_{i+1} \setminus P_i$.

Let $P(w) = P_n, Q(w) = Q_n$. Then the map $w \mapsto (P(w), Q(w))$ is called the RSK correspondence.
Remark. It is well-known that the RSK correspondence establishes a bijection between $S_n$ and the set of pairs of standard Young tableau of the same shape.

The RSK correspondence can now be used to describe the left, right and two sided cells in $S_n$. We omit the details and simply give the description. Details can be found in [Wil].

**Proposition 4.3.** For $x, y \in S_n$,
1. $x \sim_R y \iff P(x) = P(y)$.
2. $x \sim_L y \iff Q(x) = Q(y)$.
3. $x \sim_{LR} y \iff P(x)$ has the same shape as $P(y)$.

We finish by describing the cell modules.

**Proposition 4.4.** Let $C$ be a cell (left, right or two-sided determined by context). Then,
1. The left cell module associated to $C$ is the Specht module associated to the Young diagram determined by $P(x)$ for any $x \in C$.
2. The corresponding right cell module is the dual of the Specht module (viewed as a right module over the algebra $\mathcal{H}$).
3. The corresponding two sided cell module is the endomorphism algebra of the Specht module.

**References**


[Wil] Geordie Williamson. Mind your p and q-symbols, why the kazhdan-lusztig basis of the hecke algebra of type a is cellular. *Honors Theses in Pure Mathematics, University of Sydney*. 

6