1. Introduction/recap

In the previous lecture we have stated the Kac theorem and introduced the deformed preprojective algebras. In this lecture we will prove a weaker version of the theorem by studying the representation theory of those algebras.

**Theorem 1.1.** Let $Q$ be a quiver and $v$ be a dimension vector. Then the following is true.

1. If there is an indecomposable representation of dimension $v$, then $v$ is a root.
2. If $v$ is a real root, then there is a unique (up to an isomorphism) indecomposable representation of dimension $v$.
3. If $v$ is primitive (meaning that $\gcd(v_i) = 1$) and there is an indecomposable representation of dimension $v$, then $p_v$, the number of parameters for the isomorphism classes of indecomposable representations, equals $1 - (v,v)/2(= 1 - \dim G_v + \dim \text{Rep}(Q,v))$.

**Remark 1.2.** Suppose that there is $i$ such that $v_j = 0$ for all $j \neq i$ and there is no loop at $i$. Then $\text{Rep}(Q,v) = \{0\}$. The zero representation is indecomposable if and only if $v_i = 1$ (i.e., $v$ corresponds to a simple root).

Also note that if there is an indecomposable representation of dimension $v$, then the support of $v$ is connected.

Now recall that a deformed preprojective algebra is defined by

$$\Pi^\lambda(Q) = \mathbb{C} Q / (\sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \lambda_i \epsilon_i).$$

The set $\text{Rep}(\Pi^\lambda(Q), v) \subset \text{Rep}(Q, v)$ coincides with $\mu^{-1}(\sum_i \lambda_i \text{id}_{V_i})$, where $\mu: \text{Rep}(Q, v) \rightarrow g_v$ is the moment map, $\mu_i(x_a, x_{a^*}) = \sum_{a,h(a)=i} x_a x_{a^*} - \sum_{a,t(a)=i} x_{a^*} x_a$. Being a moment map means that $\mu$ is $G_v$-equivariant and

$$\langle d_x \mu(v), \xi \rangle = \omega(\xi x, v).$$

Note that $\sum_i \text{tr} \mu_i(x_a, x_{a^*}) = 0$. By $\tilde{g}_v$, we denote the subalgebra of $g_v$ consisting of all elements $(y_i)$ with $\sum_i \text{tr}(y_i) = 0$. So $\mu: \text{Rep}(Q,v) \rightarrow \tilde{g}_v$.

2. Connection to indecomposable representations of $Q$

Let $\pi: \text{Rep}(\Pi^\lambda(Q), v) \rightarrow \text{Rep}(Q,v)$ denote the projection, it sends $(x_a, x_{a^*})_{a \in Q_1}$ to $(x_a)_{a \in Q_1}$. Our goal is to describe the pre-image of $(x_a)$.
2.1. **Exact sequence.** A key tool for this is the following lemma. We define a map \( c : \text{Rep}(Q^{op},v) \to \mathfrak{g}_v \) by \( c(x_\alpha^*) := \mu(x_\alpha, x_\alpha^*) \) and a map \( t : \mathfrak{g}_v \to \text{End}(x_\alpha)^* \) by \( \langle t(y_i), (z_i) \rangle = \sum_i \text{tr}(y_i z_i) \). Recall that \( \text{End}(x_\alpha) \) denote the endomorphism algebra of the representation \( x_\alpha \), it consists of all \( Q_\alpha \)-tuples \((z_i)\) with \( z_{h(a)} x_\alpha = x_\alpha z_{l(a)} \).

**Lemma 2.1.** The sequence \( \text{Rep}(Q^{op},v) \xrightarrow{c} \mathfrak{g}_v \xrightarrow{t} \text{End}(x_\alpha)^* \to 0 \) is exact.

**Proof.** The map \( t \) is the composition of the identification \( \mathfrak{g}_v \cong \mathfrak{g}_v^* \) and the projection \( \mathfrak{g}_v^* \to \text{End}(x_\alpha)^* \), so \( t \) is surjective.

Let us prove that \( t \circ c = 0 \). This is equivalent to \( \sum_{i \in Q_0} \text{tr}(\mu(\alpha(x_\alpha, x_\alpha^*), z_i) = 0 \). But

\[
\sum_{i \in Q_0} \text{tr}(\mu_i(x_\alpha, x_\alpha^*), z_i) = \sum_{\alpha \in Q_1} (\text{tr}(x_\alpha^* x_\alpha^* z_{h(a)}) - \text{tr}(x_\alpha^* x_\alpha^* z_{l(a)})) = \\
\sum_{\alpha \in Q_1} (\text{tr}(x_\alpha^* (z_{h(a)} x_\alpha - x_\alpha^* z_{l(a)}))) = 0.
\]

In order to check that \( \ker t = \text{im} c \), we will compare the dimensions. We have

\[
\ker c = \{(x_\alpha) | \langle d_{(x_\alpha,0)} \mu((0,x_\alpha^*)) = 0 \} = \{(x_\alpha^*) | ((x_\alpha^*),\mathfrak{g}_v.(x_\alpha)) = 0 \}.
\]

The dimension of \( \mathfrak{g}_v.(x_\alpha) \) is \( \dim \mathfrak{g} - \dim \text{End}(x_\alpha) \) and so \( \dim \ker c = \dim \text{Rep}(Q^{op},v) - \dim \mathfrak{g}_v + \dim \text{End}(x_\alpha) \). We conclude that \( \dim \text{im} c = \dim \mathfrak{g}_v - \dim \text{End}(x_\alpha) = \dim \ker t \). The second equality holds because \( t \) is surjective. \( \square \)

2.2. **Consequences.** Now let us deduce some corollaries on \( \pi^{-1}(x_\alpha) \).

**Corollary 2.2.** The following is true.

1. We have \( \pi^{-1}(x_\alpha) \neq \emptyset \) if and only if \( \sum_{i \in Q_0} \lambda_i \text{tr}(z_i) = 0 \) for any \((z_i) \in \text{End}(x_\alpha)\).

2. If \( \pi^{-1}(x_\alpha) \) is non-empty, then it is an affine space of dimension \( \dim \text{Rep}(Q,v) - \dim G_v.(x_\alpha) \).

3. Suppose that \( v \) is generic with \( \lambda \cdot v = 0 \) meaning that the equality \( \lambda \cdot v' = 0 \) with \( v' \in v \) (component-wise) implies \( v = kv' \) for some \( k \in Q \) (here we write \( \lambda \cdot v = \sum_{i \in Q_0} \lambda_i v_i \)). Then \( \pi^{-1}(x_\alpha) \neq \emptyset \) if and only if the dimensions of all direct summands of \( (x_\alpha) \) are proportional to \( v \).

4. In addition, suppose \( v \) is primitive. Then \( \pi^{-1}(x_\alpha) \neq \emptyset \) if and only if \( (x_\alpha) \) is indecomposable. Moreover, all representations of \( \Pi^1(Q) \) of dimension \( v \) are irreducible.

**Proof.** (1) is a direct corollary of Lemma 2.1. (2) follows from the proof because \( \dim \pi^{-1}(x_\alpha) = \dim \ker c = \dim \text{Rep}(Q^{op},v) - \dim \mathfrak{g}_v + \dim \text{End}(x_\alpha) = \dim \text{Rep}(Q,v) - \dim G_v.(x_\alpha) \).

Let us prove (3). Let \( (x'_\alpha) \) be a direct summand of \( (x_\alpha) \) of dimension \( v' \). Let \( (z_i) \in \bigoplus \text{End}(V_i) \) denote the corresponding projection. Then it is an element of \( \text{End}(x'_\alpha) \). So \( \sum_{i \in Q_0} \lambda_i \text{tr}(z_i) = \lambda \cdot v' = 0 \). Since \( \lambda \) is generic, we see that \( v' \) is proportional to \( v \).

Conversely, let \( (x_\alpha) = \bigoplus (x'_\alpha) \) be the decomposition into indecomposables. Assume that the dimensions \( v' \) are proportional to \( v \). Let us write an endomorphism \((z_i)\) of \((x_\alpha)\) as a matrix \((z^j_i)\), with \( z^j_i \in \text{Hom}_Q((x'_\alpha),(x'_\alpha)) \). Note that since \((x'_\alpha)\) is indecomposable, the endomorphism \( z^j_i \) acts on the corresponding representation space \( V_j = \bigoplus V_i^j \) with a single eigenvalue. It follows that the vector \((\text{tr}(z^j_i))_{i \in Q_0}\) is proportional to \( v^j \). We see that \( \sum_{i \in Q_0} \lambda_i \text{tr}(z^j_i) = 0 \) and so \( \sum_{i \in Q_0} \lambda_i \text{tr}(z_i) = 0 \). (3) is fully proved.
Now let us prove (4). The first claim is a direct corollary of (3). To prove the second statement, let \((x_a', x_a'')\) be a nonzero sub of \( (x_a, x_a') \in \text{Rep}(\Pi^\lambda(Q), v)\). Then \(\pi^{-1}(x_a') \neq 0\) and hence, by (4), we need to have \(\lambda \cdot v' = 0\). Since \(v\) is primitive, this is only possible if \(v = v'\).

\[ \square \]

2.3. Application to Kac’s theorem. Let us compute \(d_v\) under some additional assumptions.

Corollary 2.3. Assume that \(v\) is primitive and \(\lambda\) is generic with \(\lambda \cdot v = 0\). If there is an indecomposable representation in \(\text{Rep}(Q, v)\) or \(\text{Rep}(\Pi^\lambda(Q), v) \neq \emptyset\), then \(p_v = 1 - (v, v)/2\).

Proof. Let \(\text{Rep}^{\text{ind}}(Q, v) \subset \text{Rep}(Q, v)\) denote the subset of the indecomposable representations. Then \(\text{Rep}(\Pi^\lambda(Q), v)\) admits a morphism \(\pi\) with image \(\text{Rep}^{\text{ind}}(Q, v)\) whose fiber over \((x_a)\) is an affine space of dimension \(\dim \text{Rep}(Q, v) - \dim \mathcal{G}_v.(x_a)\). By (4) of the previous corollary all representations in \(\text{Rep}(\Pi^\lambda(Q), v)\) are irreducible. By the Schur lemma, all their endomorphisms are constant.

Let \(\mathcal{G}_v\) denote the quotient of \(G_v\) by the one-dimensional subgroup of constant matrices. The kernel of \(G_v \to \mathcal{G}_v\) acts on \(\text{Rep}(Q, v)\) trivially so we get an action of \(\mathcal{G}_v\) on \(\text{Rep}(Q, v)\). The Lie algebra of \(\mathcal{G}_v\) is naturally identified with \(\mathfrak{g}_v\) and \(\mu : \text{Rep}(Q, v) \to \mathfrak{g}_v\) is the moment map. Also note that the action of \(\mathcal{G}_v\) on \(\text{Rep}(\Pi^\lambda(Q), v) = \mu^{-1}(\lambda)\) is free. From here and (1.1) one deduces that \(\mu\) is a submersion at all points of \(\text{Rep}(\Pi^\lambda(Q), v)\) and hence \(\dim \text{Rep}(\Pi^\lambda(Q), v) = \dim \text{Rep}(Q, v) - \dim \mathfrak{g}_v\).

Let us cover \(\text{Rep}^{\text{ind}}(Q, v)\) with locally closed \(G_v\)-stable subvarieties with constant dimensions of orbits, \(\text{Rep}^{\text{ind}}(Q, v) = \bigsqcup_i X_i,\) let \(d_i\) denote the dimension of a \(G_v\)-orbit in \(X_i\). Let \(Y_i\) denote the preimage of \(X_i\) in \(\mu^{-1}(\lambda)\), it is an affine bundle with rank \(\dim \text{Rep}(Q, v) - d_i\) over \(X_i\). So we see that \(2 \dim \text{Rep}(Q, v) - \dim \mathcal{G}_v = \max_i(\dim Y_i) = \max_i(\dim X_i + \dim \text{Rep}(Q, v) - d_i) = \dim \text{Rep}(Q, v) + \max_i(p(X_i))\). It follows that \(p_v = \max_i(p(X_i)) = \dim \text{Rep}(Q, v) - \dim \mathcal{G}_v + 1 = 1 - (v, v)/2\).

\[ \square \]

3. Reflection functors

We will view \(\lambda = (\lambda_j)_{j \in Q_0}\) as an element of \(\mathfrak{h}\) and a dimension vector \(v\) as an element of \(\mathfrak{h}^*\) (the pairing is by \((\lambda, v) = \lambda \cdot v\)). Recall that \(W(Q)\) acts on \(\mathfrak{h}^*\) as follows: \((s_i v)_{j} = v_j\) for \(j \neq i\) and \((s_i v)_{i} = \sum_{j} n_{ij} v_{j} - v_{i}\), where \(n_{ij}\) is the number of edges between \(i\) and \(j\). So \(W(Q)\) acts on \(\mathfrak{h}\) as follows: \((s_i \lambda)_{j} = -\lambda_i, (s_i \lambda)_{j} = \lambda_j + n_{ij} \lambda_i\).

The main result of this section is as follows.

Theorem 3.1. Pick \(i \in Q_0\) such that there are no loops at \(i\). Suppose \(\lambda_i \neq 0\). Then is an equivalence \(\Pi^\lambda(Q)\)-mod \(\to \Pi^{s_i \lambda}(Q)\)-mod that maps a representation of dimension \(v\) to a representation of dimension \(s_i v\).

Before proving this theorem we will explain how it applies to the Kac theorem.

3.1. Application to Kac’s theorem.

Corollary 3.2. Suppose there is an indecomposable representation of dimension vector \(v\). Then \(v\) is a root.

Proof. We can assume that for all \(v' \leq v, v' \neq v\) (componentwise), the claim is true. We can also assume \((v, v) > 0\), otherwise we are done by Remark 1.2. If \((v, e_i) \leq 0\) for all \(i\), then \((v, v) = \sum_i v_i (v, e_i) \leq 0\). Note that if there is a loop at \(i\), then \((v, e_i) \leq 0\). So it’s enough
to consider the case when there is $i$ such that there is no loop at $i$ and $(v, \epsilon_i) > 0$ so that $s_i v = v - (v, \epsilon_i) \epsilon_i < v$.

Let us prove that if $\text{Rep}(\Pi^\lambda(Q), v)$ for a Zariski generic $\lambda$ with $\lambda \cdot v = 0$ contains an indecomposable representation, then $v$ is a real root. We prove it by induction. By Theorem 3.1, $\text{Rep}(\Pi^{s_i \lambda}(Q), s_i v)$ contains an indecomposable representation. This provides an inductive step. The base is given by $v = m \epsilon_i$; there the representation is zero and so $m = 1$.

By (3) of Corollary 2.2, if $\text{Rep}(Q, v)$ contains an indecomposable representation, then so does $\text{Rep}(\Pi^\lambda(Q), v)$. This completes the proof. \hfill $\square$

**Corollary 3.3.** Let $v$ be a real root. Then there is a unique (up to isomorphism) indecomposable representation of $Q$ with dimension vector $v$.

**Proof.** Let $\lambda$ be generic with $\lambda \cdot v = 0$. Let us check that there is a unique (up to an isomorphism) representation of $\Pi^\lambda(Q)$ of dimension vector $v$. If $v = \epsilon_i$, then there is only the zero representation and so we are done. Theorem 3.1 gives the induction step.

Note that, as any real root, $v$ is indecomposable. Now the claim of this corollary follows from (4) of Corollary 2.2. \hfill $\square$

### 3.2. Construction of equivalence

Now let us construct the required equivalence. Pick a representation $(x_a, x_a^*)$ with dimension vector $v$. Recall that $\Pi^\lambda(Q)$ does not depend on the orientation of $Q$ up to an isomorphism. So we may assume that $i$ is a sink in $Q$. Let $W_i := \bigoplus_{a, t(a) = i} V_{h(a)}$. We can write $(x_a, x_a^*)$ as $(A, B, x)$, where $A := \bigoplus_{a, t(a) = i} : V_i \to W_i, B := \bigoplus_{a, t(a) = i} x_a^* : W_i \to V_i$ and $x$ includes all $x_b, x_b^*$ with $t(b) \neq i$. Multiplying the relation of $\Pi^\lambda(Q)$ by $\epsilon_i$, we see that $BA = -\lambda_i \text{id}_{W_i}$. Since $\lambda_i \neq 0$, we see that $A$ is injective, $B$ is surjective. Also, we see that $W_i = \text{im} A \oplus \text{ker} B$. Identifying $V_i$ with $im A$, we can assume that $A$ is the inclusion $V_i \hookrightarrow W_i$ and $B = -\lambda_i \pi$, where $\pi$ is the projection along $\text{ker} B$.

Now let us proceed to defining a representation of $\Pi^{s_i \lambda}(Q)$ with dimension vector $s_i v$. The space $V' := \bigoplus V_i'$ is determined as follows: $V'_j := V_j$ if $j \neq i$, and $V'_i := \text{ker} B$. In particular, $v' = s_i v$. The representation is given by $(A', B', x)$, where $A'$ is the inclusion $V_i' \hookrightarrow W_i$ and $B'$ is $\lambda_i \pi'$, where $\pi' : W_i \to V_i'$ is the projection along $A$. Note that we have

$$\tag{3.1} A' B' - AB = \lambda_i \text{id}_{W_i}. $$

Now let us check that the resulting representation $(A', B', x)$ factors through $\Pi^{s_i \lambda}(Q)$. For $a \in Q_i$ with $t(a) = i$, let $\rho_a, t_a$ denote the projection $W_i = \bigoplus_{a, t(a) = i} V_{h(a)} \to V_{h(a)}$ and the inclusion $V_{h(a)} \to W_i$ corresponding to this arrow. So we have $x_a = \rho_a \circ A, x_a^* = B \circ t_a, x_a' = \rho_a \circ A', x_a'^* = B' \circ t_a$. We have $- \sum_{t(a) = i} x_a'^* x_a = -B' A' = -\lambda_i \text{id}_{V_i'}$. So what we need to check is that for $j \neq i$, we have

$$\sum_{a, h(a) = j} x_a'^* x_a^* - \sum_{a, t(a) = j} x_a'^* x_a = (s_i \lambda)_j \text{id}_{V_j} = (\lambda_j + n_{ij} \lambda_i) \text{id}_{V_j}. $$

This will follow if we check that

$$\sum_{a, h(a) = j} (x_a'^* x_a^* - x_a x_a^*) - \sum_{a, t(a) = j} (x_a'^* x_a^* - x_a x_a^*) = n_{ij} \lambda_i \text{id}_{V_j}. $$


If \( t(a) \neq i \), then \( x_a = x'_a, x_{a^*} = x'_{a^*} \). So the left hand side is

\[
\sum_{t(a)=i, h(a)=j} (x'_a x'_{a^*} - x_a x_{a^*}) = \sum_{t(a)=i, h(a)=j} \rho_a \circ (A'B' - AB) \circ \tau_a = \\
= [(3.1)] = \sum_{t(a)=i, h(a)=j} \rho_a \circ (\lambda_i \text{id}_{W_i}) \circ \tau_a = n_{ij} \lambda_i \text{id}_{V_j},
\]
as required. So we indeed get a a representation of \( \Pi_{s,\lambda}(Q) \).

Our construction is functorial. Indeed, let \((y_i): (V_i, x_a, x_{a^*}) \to (\tilde{V}_i, \tilde{x}_a, \tilde{x}_{a^*})\) be a homomorphism of representations. This induces a homomorphism \( y: W_i \to \tilde{W}_i \) that intertwines \( A, B \) with \( \tilde{A}, \tilde{B} \). In particular, \( y \) restricts to \( \ker B \to \ker \tilde{B} \). So it induces a homomorphism \((y'_i): (V'_i, x'_a, x'_{a^*}) \to (\tilde{V}_i, \tilde{x}'_a, \tilde{x}'_{a^*})\). We indeed get a functor \( \Pi^\lambda(Q) \text{-mod} \to \Pi^{s,\lambda}(Q) \text{-mod} \) that behaves as \( s \) on the dimension vectors.

We also have a similarly defined functor \( \psi: \Pi^{s,\lambda}(Q) \text{-mod} \to \Pi^\lambda(Q) \text{-mod} \). It sends a representation \((A', B', \bar{x})\) back to \((A, B, x)\). It is easy to see that \( \psi \circ \varphi \) is isomorphic to the identity functor of \( \Pi^\lambda(Q) \text{-mod} \). Similarly, \( \varphi \circ \psi \) is isomorphic to the identity functor. This shows that \( \varphi \) is an equivalence (with quasi-inverse \( \psi \)).

4. Further results and applications

4.1. Irreducible representations. A basic question about the representation theory of \( \Pi^\lambda(Q) \) is to describe its irreducible representations. Let us state the corresponding result of Crawley-Boevey, we are not going to provide a proof.

For \( v \in \mathbb{Z}_{\geq 0} \), set \( p(v) = 1 - \frac{1}{2}(v, v) \). Define the set \( \Sigma_\lambda \) of all positive roots such that \( \lambda \cdot v = 0 \) \( p(v) > \sum_{i=1}^k p(v_i) \) for all proper decompositions of \( v \) into the sum \( \sum_{i=1}^k v_i \), where all \( v_i \in (\mathbb{Z}_{\geq 0})^{Q_0} \setminus \{0\} \) such that \( \lambda \cdot v_i = 0 \). It is not so easy to describe \( \Sigma_\lambda \), but this is a combinatorial object.

**Theorem 4.1.** The algebra \( \Pi^\lambda(Q) \) has an irreducible representation of dimension \( v \) if and only if \( v \in \Sigma_\lambda \). Moreover, \( \text{Rep}(\Pi^\lambda(Q), v) \subset \text{Rep}(\mathbb{C}Q, v) \) is an irreducible subvariety of dimension \( \dim \text{Rep}(\Pi^\lambda(Q), v) + p(v) \) and a Zariski generic point in \( \text{Rep}(\Pi^\lambda(Q), v) \) gives an irreducible representation.

4.2. Application to additive Deligne-Simpson problem. The additive Deligne-Simpson problem (we’ll abbreviate this as DS problem) asks about the conditions on the conjugacy classes \( C_1, \ldots, C_k \) in \( \text{Mat}_n(\mathbb{C}) \) such that there are matrices \( Y_i \in \text{Mat}_n(\mathbb{C}) \) satisfying the following two conditions:

1. \( \sum_{i=1}^k Y_i = 0 \),
2. and there are no proper subspaces in \( \mathbb{C}^n \) stable under all \( Y_i \).

From \( C_1, \ldots, C_k \), Crawley-Boevey have constructed a quiver \( Q \), a dimension vector \( v \), and \( \lambda \in \mathbb{C}^{Q_0} \) such that there is a bijection between

1. solutions \((Y_1, \ldots, Y_k)\) of the DS problem (up to \( \text{GL}_n(\mathbb{C}) \)-conjugacy),
2. irreducible dimension \( v \) representations of \( \Pi^\lambda(Q) \) (up to an isomorphism).

Then the solution of the DS problem follows from Theorem 4.1 (one needs to use some complicated combinatorics to get the answer explicitly).