1 Introduction and Disclaimer

We will sketch the computation by Maulik and Okounkov of the quantum cohomology of $\text{Hilb}_n \mathbb{C}^2$.

As you will see, the proof is somewhat indirect, but the methods used apply to general quiver varieties, and yield a variety of other great results. See [3] for a more direct proof. Due to limitations in space and time, we will limit ourselves to a very brief overview and gloss over all technical points, and many statements will only be true up to a sign. We will not state results at their natural level of generality.

For the above reasons, do not use this chapter as a technical reference! It is only an appetizer for the main course: Quantum Groups and Quantum Cohomology, by Maulik and Okounkov [1].

2 Reminder on quantum cohomology and the Steinberg algebra

For notes, refer to the previous semester.

3 Main Result

$H^2(\text{Hilb}_n \mathbb{C}^2)$ is 1-dimensional, and generated by $c_1(\mathcal{V})$. We will sketch a proof of the following

Theorem 3.1 Quantum multiplication by the divisor $c_1(\mathcal{V}) \in H^2(\text{Hilb}_n \mathbb{C}^2)$ is given by

$$c_1(\mathcal{V}) + \hbar \sum_{k>0} \frac{kq^k}{1-q^k} \alpha_k \alpha_{-k} - \hbar \frac{q}{1-q} \sum_{n>0} \alpha_{-n} \alpha_n$$

The left hand summand indicates the classical cup product by $c_1(\mathcal{V})$. The rightmost term acts by a scalar on any fixed Hilbert scheme; the reader may safely ignore it for now.

Breaking from the notation of the previous talk, we will write $\alpha_k, \alpha_{-k}$ for the action of the Heisenberg algebra elements $Z_{pt}[k], Z_1[-k]$ respectively. If one desires an expression purely in terms of Heisenberg operators, one can use Lehn’s formula from the previous lecture to rewrite the cup product by $c_1(\mathcal{V})$.

Note that this is formally similar to the result for the Springer resolution, with the lie algebra $g$ replaced by the Heisenberg algebra.

4 Guiding principle for the proof

As we saw last semester, the quantum cohomology of the Springer resolution is a commutative subalgebra of the non-commutative algebra generated by Steinberg correspondences and characteristic classes, i.e. the graded affine Hecke algebra.

The Hilbert scheme is also a symplectic resolution, and one can similarly argue that its quantum cohomology is a commutative subalgebra of a non-commutative, non-cocommutative Hopf algebra $Y$, a kind of ‘Yangian of the Heisenberg algebra’. $Y$ contains the Heisenberg algebra, and acts on the Fock
space representation $V$ given by the union of all Hilbert schemes (described in
the previous talk).

One may think of $V$ as the ‘basic’ representation of $Y$. In fact it depends on
a parameter $a \in \mathbb{C}$, and we write it as $V(a)$. Our first step is to construct the
tensor products $V(a_1) \otimes ... \otimes V(a_r)$ geometrically. One can then play the tensor
structure against the quantum product to determine the quantum cohomology
of $\text{Hilb}_n \mathbb{C}^2$.

Though we will not use (or define) $Y$ explicitly, we will occasionally refer to
it as a guiding idea. We will italicize references to $Y$ to underline their purely
heuristic nature.

5 The moduli of framed sheaves

In this section, we construct the spaces $\mathcal{M}(r)$ whose cohomologies are the tensor
products of the basic representation of $Y$.

We constructed $\text{Hilb}_n \mathbb{C}^2$ by symplectic reduction:

$$\text{Hilb}_n \mathbb{C}^2 = T^*(\text{End}(\mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^1, \mathbb{C}^n))//_0 \text{Gl}(n).$$

We can similarly define

$$\mathcal{M}(r,n) = T^*(\text{End}(\mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^n))//_0 \text{Gl}(n).$$

$\mathcal{M}(r,n)$ parametrizes stable rank $r$ framed torsion free sheaves on $\mathbb{P}^2$, with
class $\mathcal{F} = n$. We will not use this interpretation, and refer readers to [2] for a
proof. More importantly for our purposes, it is also a symplectic resolution.

As with the Hilbert scheme, the $\mathbb{C}^*$ action on the cotangent fibers descends
to an action on $\mathcal{M}(r,n)$ dilating the symplectic form by a character $\hbar$. We will
write this torus as $\mathbb{C}^*_\hbar$, and we will usually work over the field of fractions
$k = H^*_C(pt)_{\text{frac}}$. The action of $A = (\mathbb{C}^*)^r$ on $\mathbb{C}^r$ induces a symplectic action
on $\mathcal{M}(r,n)$. Set $G = A \times \mathbb{C}^*_\hbar$.

Set

$$\mathcal{M}(r) = \bigcup_{n=0,1,2,...} \mathcal{M}(r,n).$$

Note that the same torus $A$ acts on each component. The Heisenberg algebra
acts on $H^*_G(\mathcal{M}(r))$ (in much the same way it acted on the union of hilbert
schemes); we write $\alpha_k, \alpha_{-k}$ for the action of the generators.

Finally, the vector space $\mathbb{C}^n$ descends to a tautological bundle $V$ on $\mathcal{M}(r)$,
which coincides with the usual one on the Hilbert scheme.

6 Tensor structure

In this section we construct the map intertwining the representation $H^*_G(\mathcal{M}(r))$
with $V(a_1) \otimes ... \otimes V(a_r)$.

The following is left as an exercise to the reader.
Proposition 6.1
\[ \mathcal{M}(r, n)^A = \bigcup_{n_1 + \ldots + n_r = n} \prod_{i=1}^{r} \mathcal{M}(1, n_i) \]

We can write this more concisely as

**Proposition 6.2**
\[ \mathcal{M}(r)^A = \mathcal{M}(1)^r \]

Using the Kunneth formula, one obtains
\[ H_*^G(\mathcal{M}(r)^A) = H_*^{C^*}(\mathcal{M}(1)) \otimes H_*^A(pt) \]

where the tensor product is taken over \( k \). The localization theorem thus provides an isomorphism of localized cohomologies
\[ H_*^G(\mathcal{M}(r))_{loc} \cong H_*^{C^*}(\mathcal{M}(1)) \otimes H_*^A(pt)_{loc}. \]

However, this is not the map we want. We will describe a different, unlocalized and degree-preserving map called the stable envelope,
\[ H_*^C(\mathcal{M}(1)) \otimes H_*^A(pt) = H_*^G(\mathcal{M}(r)^A) \xrightarrow{\text{Stab}_C} H_*^G(\mathcal{M}(r)), \]

which depends on an ordering \( C \) of the factors to the left. It corresponds to the intertwiner with the corresponding ordered product of the \( V(a_i) \).

### 6.1 Inductive definition of \( \text{Stab}_C \)

For simplicity we restrict our discussion to \( \mathcal{M}(2) \). Since the action of \( A \) factors through its subtorus \( B = \{(z, w) : zw = 1\} \subset A \), we will often implicitly use the former rather than the latter. Set \( C^+ = [1, 2] \) and \( C^- = [2, 1] \). Consider the corresponding coweights
\[ \sigma_+ : C^* \to B, z \to (z, z^{-1}) \]
\[ \sigma_- : C^* \to B, z \to (z^{-1}, z) \]

Let \( \gamma \in H_*^G(X^A) \) be represented by a geometric cycle \( \hat{\gamma} \). Let
\[ \text{Leaf}(\gamma) = \{ x \in \mathcal{M}(2) : \lim_{z \to 0} \sigma_+(z) \cdot x \in \hat{\gamma} \}. \]

To a first approximation, \( \text{Stab}_{C^+}(\gamma) = \overline{\text{Leaf}(\gamma)} \). However, this cycle may intersect other fixed loci. The actual stable basis minimizes such intersections, in the following sense.

Given two fixed loci \( Z_1 \) and \( Z_2 \), we say \( Z_1 \geq Z_2 \) if \( \overline{\text{Leaf}(Z_1)} \) intersects \( Z_2 \). The transitive closure of this relation defines a partial ordering of the fixed loci. Define
\[ \text{Slope}(Z) = \bigcup_{Z' \leq Z} \overline{\text{Leaf}(Z')} \]

Note that
\[ \mathcal{M}(n_1, 1) \times \mathcal{M}(n - n_1, 1) = Z_1 \geq Z_2 = \mathcal{M}(n_2, 1) \times \mathcal{M}(n - n_2, 1) \]
iff $n_1 \leq n_2$.

A acts trivially on any component $K$ of $\mathcal{M}(r)^A$, hence we have a (non-canonically) isomorphism

$$H^*_G(K) \xrightarrow{\sim} H^*_C(K) \otimes H^*_A(pt).$$

Given $\gamma \in H^*_G(K)$, we define its 'A-degree'

$$\deg_A(\gamma) \in \mathbb{N}$$

as the highest degree occurring in the RHS factor; it does not depend on the choice of isomorphism. Let $Z_1, Z_2$ be components of $\mathcal{M}(r)^A$. The normal bundle to a component $Z$ splits as a sum of dilating and contracting directions under $A$:

$$N_Z = N^+_Z \oplus N^-_Z.$$  

Since $A$ is symplectic, $\dim N^+_Z = \dim N^-_Z = \frac{1}{2} \text{codim} Z$. Now let $\gamma \in H^*_C(Z_1)$. Let $\iota_j : Z_j \to \mathcal{M}(r)$ be the inclusions.

**Theorem 6.3** There exists a unique $H^*_G(pt)$-linear map

$$\text{Stab}_+ : H^*_G(\mathcal{M}(2)^A) \to H^*_G(M(2))$$

satisfying the following requirements. For $\gamma \in H^*_C, (\mathcal{M}(2))$,

$$\iota^*_j \text{Stab}_+(\gamma)) = eu(N^+_Z)\gamma$$

$$\deg_A(\iota^*_j \text{Stab}_+(\gamma)) < \frac{1}{2} \text{codim} Z_2$$

$\text{Stab}_+(\gamma)$ is supported on the slope of $Z_1$

This is achieved essentially by taking the intersection of $\gamma_2 = \text{Leaf}(\gamma) \cap Z_2$, adding some multiple of $\text{Leaf}(\gamma_2)$ to $\text{Leaf}(\gamma)$, and proceeding inductively.

The above properties of $\text{Stab}_+(\gamma)$ ensure that its restriction to other fixed loci have low $A$-degree. We will often take a limit in the equivariant parameters for which such contributions vanish.

**Example** We have $\mathcal{M}(1, 2) = T^*\mathbb{P}^1 \times \mathbb{C}^2$. We have $A = (\mathbb{C}^*)^2$, acting by rotations on the first factor and trivially on the second. We have

$$\mathcal{M}(1, 2)^A = \mathcal{M}(1, 1) \times \mathcal{M}(0, 1) \cup \mathcal{M}(0, 1) \times \mathcal{M}(1, 1) = \mathbb{C}^2 \times [0, 1] \cup \mathbb{C}^2 \times [1, 0].$$

We now drop the factors of $\mathbb{C}^2$; the diligent reader can insert them back in. Let $\gamma_0 \in H^*_C([0, 1])$ and $\gamma_1 \in H^*_C([1, 0])$ be the fundamental classes. Let $L_0$ be the zero-section of $T^*\mathbb{P}^1$, and let $L_1$ be the fiber above $[1, 0]$.

Then $\text{Stab}_+(\gamma_0) = L_0 + L_1$, and $\text{Stab}_+(\gamma_1) = L_1$. Choosing $\text{Stab}_-\text{reverses}$ the roles of the two fixed points.
6.2 $\text{Stab}_+ \text{ from an affine deformation}$

Here is an alternative construction of the stable basis. The reader may skip this part if he or she wishes.

First, we deform $\mathcal{M}(r)$ to the affine space

$$\mathcal{M}(r)_\lambda = T^*(\text{End}(\mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)) / \lambda^* \text{Gl}(n).$$

where $\lambda \in \mathbb{C}$. For $\lambda \neq 0$, this is a smooth affine space; when $r = 1$ it is the phase space of the ‘rational Calogero-Moser’ system. $\mathbb{C}^*_\hbar$ acts on the total space of this deformation, preserving only the fiber at $\lambda = 0$, whereas $A$ acts fiberwise.

Let $\mathcal{M}(2)_{A^1 \backslash 0}$ be the total space of the deformation away from $\lambda = 0$. Consider the smooth, closed $G$-stable subvariety

$$L \subset \mathcal{M}(r)_{A^1 \backslash 0} \times \mathcal{M}(r)_{A^1 \backslash 0}$$

consisting of pairs $(x, y)$ such that $y$ flows to $x$ under $\sigma_+$. Define

$$\text{Stab}_+ \subset \mathcal{M}(2)^A \times \mathcal{M}(r)$$

to be the intersection of the closure of $L$ with the fiber at $\lambda = 0$. The resulting correspondence defines $\text{Stab}_+$.

7 R-matrix

From now on, we write $V = H^*_\mathbb{C},(\mathcal{M}(1))$ for brevity.

Define the ‘R-matrix’

$$R(u) = \text{Stab}_-^{-1} \circ \text{Stab}_+.$$ 

Here $u$ is the equivariant parameter of the torus $B$. $R(u)$ is a $\mathbb{C}(u)$-linear automorphism of $V \otimes^2 \otimes \mathbb{C}(u)$. Using an analogous definition for $r = 3$, one can easily check that it satisfies the spectral Yang-Baxter equation, and hence can be used to define the aforementioned ‘Yangian’ $Y$ acting on the cohomology of the Hilbert scheme.

We will not pursue that route: instead we enumerate a few properties of $R(u)$. Let $\text{Stab}_+^\tau$ be the transposed correspondence, going from $X$ to $X^A$.

**Theorem 7.1**

$$\text{Stab}_-^{-1} = \text{Stab}_+^\tau.$$ 

This can be proven as follows: One shows that the composition $\text{Stab}_+^\tau \circ \text{Stab}_-$ involves only proper maps, hence we may specialize equivariant parameters as we please. Using the localization theorem, it is possible to express it as a composition of correspondences between fixed loci

$$\mathcal{M}(2)^A \xrightarrow{(\text{Stab}_+^\tau)^A} \mathcal{M}(2)^A \xrightarrow{(\text{Stab}_-)^A} \mathcal{M}(2)^A$$


$V$ is the direct sum of cohomologies of the components of the fixed locus. The terms of $\text{Stab}_+^A$ which are not block-diagonal in this decomposition have small $A$-degree, by the definition of the stable basis. Taking the $A$-equivariant parameters to infinity, all non block-diagonal contributions vanish. The diagonal contributions are easily seen to give the identity.

Using equivariant localization, one can also write $R(u)$ as a composition

$$\mathcal{M}(2)^A \xrightarrow{\text{Stab}_+^A} \mathcal{M}(2)^A \xrightarrow{(\text{Stab}_+^A)^A} \mathcal{M}(2)^A$$

Expanding in powers of $\frac{1}{u}$, we obtain

**Theorem 7.2**

$$R(u) = 1 + \frac{h}{u} r + O\left(\frac{1}{u^2}\right)$$

where $r$ is a Steinberg operator. We call it the classical $r$-matrix.

Using a similar properness argument, one can show

**Theorem 7.3** $R(u)$ commutes with all Steinberg operators.

In particular it commutes with the action of Heisenberg.

Let $Z = \mathcal{M}(0,1) \times \mathcal{M}(1) \subset \mathcal{M}(2)$. Its connected components are maximal with respect to the partial order on fixed loci. It follows that the restriction $R(u)^Z$ of the R-matrix to the cohomology of $Z$, i.e. $V_0 \otimes V$, has a simple form. Let $N_Z$ be the normal bundle to $Z$. Let $N_Z^+$ and $N_Z^-$ be the subbundles of directions contracted and dilated by $\sigma$, respectively.

**Theorem 7.4**

$$R(u)^Z = \frac{eu(N_Z^+ \otimes \mathcal{O})}{eu(N_Z^-)}$$

where $\mathcal{O}$ (abusively) denotes the trivial line bundle with weight $h$ under the $\mathbb{C}^*$ action.

One easily checks that that $N_Z^+ = V$ where $V$ is the tautological bundle on the second factor. Note that the euler classes involved are equivariant with respect to $A$, and the RHS is a series in $\frac{1}{u}$.

**Theorem 7.5** $R(u)$ is uniquely determined by its values on $V_0 \otimes V$ and the fact that it commutes with all Steinbergs.

The full operator $R(u)$ is quite complicated, but we can use the above theorem to show

**Theorem 7.6**

$$r = 1 \otimes N + N \otimes 1 + \sum_{k \neq 0} \alpha_{-k} \otimes \alpha_k. \quad (1)$$

where $N$ acts by multiplication by $n$ on $H^*_C(\mathcal{M}(n,1))$. 


8 R-matrix as a shift operator

Recall from last semester that for the Springer resolution $X$, one can construct 'shift operators' $S(s, q)$, for $s$ in the coweight lattice of $G$, which intertwine the quantum connection of $X$ for shifted values of the corresponding equivariant parameter:

$$S(s, q) \nabla(a) = \nabla(a + s) S(s, q).$$

Such shift operators can be defined naturally in terms of certain curve counts over an $X$-bundle over $\mathbb{P}^1$, following work of Seidel. One can similarly define shift operators for $\mathcal{M}(r)$, which shift the equivariant parameters of $G = \mathbb{C}^* \times A$.

One can organize these curve counts in such a way as to show

**Theorem 8.1** The operator $S(\sigma, q)$ is equal to quantum multiplication by some class $\gamma_\sigma \in QH_G^*(\mathcal{M}(r))$. It therefore commutes with quantum multiplication by any class.

Again using a properness argument, one can show that shift operators for the symplectic torus $A$ are expressible in terms of the R-matrix from the previous section. In particular,

**Theorem 8.2**

$$\text{Stab}^{-1} \circ S(\sigma, q) \circ \text{Stab} = q^{1 \otimes N} R(u)$$

Here $q^{1 \otimes N}$ is the operator which equals the constant $n$ on $V_k \otimes V_n$.

We can now combine the two theorems above to deduce the main result of this section. We write $Q$ for the operator of quantum multiplication on either $\mathcal{M}(1)$ or $\mathcal{M}(2)$, sometimes with a subscript $Q_r, r = 1, 2$ when helpful. Write $\Delta Q$ for the operator

$$\text{Stab}^{-1} \circ Q_2 \circ \text{Stab}$$

acting on $V \otimes V \otimes \mathbb{C}(u)$.

**Theorem 8.3**

$$[q^{1 \otimes N} R(u), \Delta Q] = 0.$$  (3)

9 Computing the quantum product

We want to compute quantum multiplication by the divisor $c_1(V)$ in $\text{Hilb}_n \mathbb{C}^2 = \mathcal{M}(n, 1)$. We proceed (roughly) by computing the coproduct of $Q$, and checking that $Q$ is determined by its coproduct.

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which corresponds to a connected component of $\mathcal{M}(2)$. One can further split it into a sum of components

$$\Delta_k Q : V_{n_1} \otimes V_{n_2} \rightarrow V_{n_1+k} \otimes V_{n_2-k}.$$ 

Using equation (4) together with a properness argument for the quantum corrections, one shows

$$\Delta_0 Q = Q \otimes 1 + 1 \otimes Q$$

(5)

This determines $Q$ from $\Delta Q$. We can determine the other components of $\Delta Q$ explicitly. Recall theorem (3):

$$[q^k \otimes R(u), \Delta Q] = 0.$$ 

The commutator of the classical part with $R(u)$ is encoded in 4. The quantum corrections, which we write as $Q^{\text{corrections}}$, are Steinberg operators, thus commute with $R(u)$. A dash of arithmetic gives

$$\sum_k (1 - q^k) \Delta_k Q^{\text{corrections}} = \hbar \sum_k (\alpha_k \otimes \alpha_{-k} - \alpha_{-k} \otimes \alpha_k)$$

This determines $\Delta_k Q$ for all $k$ except 0. Finally, we recall the conjectured expression for $Q$

$$Q_{\text{conj}} = c_1(V) + \hbar \sum_{k > 0} \frac{kq^k}{1 - q^k} \alpha_k \alpha_{-k} + \text{scalar term}.$$ 

We will show it holds on $\mathcal{M}(2)$, then deduce from (5) that it holds on $\mathcal{M}(1)$. Let

$$E = Q - Q_{\text{conj}}$$

be the error term. $E$ clearly preserves $V_0$. Note that $E$ makes sense on both $\mathcal{M}(1)$ and $\mathcal{M}(2)$. Using the results above, one computes

$$\Delta E = E \otimes 1 + 1 \otimes E.$$ 

In particular, $\Delta E$ preserves $Z = V_0 \otimes V$. It is not hard to see that $E$ is a Steinberg correspondence, whence

$$[R(u), \Delta E] = 0.$$ 

Specializing to $Z$, we get

$$[R(u)^Z, 1 \otimes E] = 0.$$ 

An operator which commutes with $R(u)^Z$ commutes with all characteristic classes of $V$.

Using a resolution of the diagonal, one can show that characteristic classes generate the localized equivariant cohomology of $\mathcal{M}(r)$. It follows that $E$ is cup product by a cohomology class. Since $E$ has cohomological degree 0, it must be a scalar.

QED.
References

