In these notes we introduce the crystal structures of modules over Kac-Moody algebras obtained from Berenstein-Kazhdan perfect bases, especially on the complexified Grothendieck groups of type A Kac-Moody categorifications. In Section 1 we describe the structure of simple objects in an $\mathfrak{sl}_2$-categorification. In Section 2, we introduce the Berenstein-Kazhdan perfect bases of integrable highest weight representations of a Kac-Moody algebra. Finally in Section 3, we apply what we have in the first two sections to the example of categorical $\mathfrak{sl}_2$-action on modules over cyclotomic Hecke algebras, and conclude that this is a categorification of an irreducible $\mathfrak{sl}_2$-module.

1. Simple objects in an $\mathfrak{sl}_2$-categorification

1.1. Reminder and notation. Let $\mathcal{C}$ be a general artinian and noetherian $\mathbb{F}$-linear abelian category equipped with a categorical $\mathfrak{sl}_2$-action given by the endofunctors $E$ and $F$, the parameter $q \in \mathbb{F}^\times$ and $a \in \mathbb{F}$, where $a \neq 0$ if $q \neq 1$, and $L \in \text{End}(E)$, $T \in \text{End}(E^2)$. We adopt some notation from [Si] and [CR]:

- Let $[\mathcal{C}] = K_0(\mathcal{C}) \otimes \mathbb{C}$ denote the complexified Grothendieck group of $\mathcal{C}$ and $H_q(n)$ denote the affine Hecke algebra generated by $X_1, \ldots, X_n, T_1, \ldots, T_n$ subject to the Hecke relations.
- For some $U \in \mathcal{C}$, denote $h_+(U) := \max \{ j : E^j U \neq 0 \}$, $h_-(U) := \max \{ j : F^j U \neq 0 \}$, and $d(U) := h_+(U) + h_-(U) + 1$. Also, denote the socle of $U$ by $\text{soc}(U)$, which is the maximal semisimple subobject of $U$ in $\mathcal{C}$, and the head by $\text{head}(U)$, which is the maximal semisimple quotient.
- $E(i)$, $F(i)$ denote the categorified divided powers.
- Let $\mathfrak{m}_n \subseteq P_n := \mathbb{F}[X_1^\pm, \ldots, X_n^\pm]$ be the ideal generated by $(X_i - a)$, $i = 1, \ldots, n$. Let $n_n := \mathfrak{m}_n^{\mathfrak{m}_n} \subseteq H_q(n)$. Let $N_n$ be the category of $H_q(n)$-modules with locally nilpotent $n_n$-action. Since $n_n$ is contained in the center of $H_q(n)$, the quotient $\overline{H}(n) = H_q(n)/n_n H_q(n)$ is an algebra. For $0 \leq i \leq n$, denote by $B_i,n$ the image of the subalgebra $H_q(i)$ inside $\overline{H}(n)$. Define the Kato modules $K_n := H_q(n) \otimes_{P_n} \mathfrak{m}_n \cong (H_q(n)/n_n)c_n$ to be the unique simple module in $N_n$, where $c_n = \sum_{w \in S_n} q^{-\ell(w)} T_w$ for $\tau \in \{ \text{triv}, \text{sign} \}$.
- As in [Si] Proposition 3.3, for any $U \in \mathcal{C}$ and $n > 0$, $E^n(U)$ has a natural left $H_q(n)$-module structure. It induces a morphism $\gamma_n : H_q(n) \to \text{End}(E^n)$, which is an isomorphism $B_i,n \cong \text{End}(E^n)$. Moreover, it induces an isomorphism $B_i,n \cong \text{End}(E^n)$.
- Given $d \geq 0$, let $\mathcal{C} \leq d$ be the Serre subcategory of $\mathcal{C}$ consisting of all simple objects $S$ such that $d(S) \leq d$. Clearly $[\mathcal{C} \leq d] \subseteq [\mathcal{C}]_{\leq d}$.

1.2. Simples in $\mathcal{C}$. In this subsection, we focus on the categorical action of $E$ and $F$ on a simple object $S$ in $\mathcal{C}$. In general, $E S$ and $F S$ (or more generally, $E(i) S$ and $F(i) S$) are not necessarily simple, but their socles and heads are.

Also we prove some results describing $\text{End}(E(i) S)$.

The following result is due to Chuang-Rouquier [CR] Proposition 5.20.

Proposition 1.1. Let $S$ be a simple object of $\mathcal{C}$, and let $n = h_+(S)$. Then, for every $i \leq n$:

(a) The object $E(i) S$ is simple.
(b) The socle and the head of $E(i) S$ are isomorphic to a simple object $S'$ of $\mathcal{C}$. We have $H_q(i)$-equivariant $\mathcal{C}$-isomorphisms: $\text{soc}(E(i) S) \cong \text{head}(E(i) S) \cong S' \otimes K_i$.
(c) The canonical homomorphism $\gamma_i(S) : H_q(i) \to \text{End}_\mathcal{C}(E(i) S)$ factors through $B_i,n$. Moreover, it induces an isomorphism $B_i,n \cong \text{End}_\mathcal{C}(E(i) S)$.

(d) We have $[E(i)(S)] - (n_i[S']) \in [\mathcal{C}]_{\leq d(S')}^{-1}$.

The corresponding statements with $E$ replaced by $F$ and $h_+(S)$ by $h_-(S)$ hold as well.

To prove the proposition, we need the following two lemmas.
Lemma 1.2. Let $M$ be an object of $C$. If $d(S) \geq r$ for any simple subobject (resp. quotient) $S$ of $M$, then $d(S') \geq r$ for any simple subobject (resp. quotient) of $EM$ or $FM$.

Proof. By the weight decomposition of $C$ ([Sl Proposition 3.5]), it is enough to consider the case where $M$ lies in a single weight space. Let $T'$ be a simple submodule of $EM$, by adjunction, $\text{Hom}(FT, M) \cong \text{Hom}(T, EM) \neq 0$. So there exists $S$ being a simple submodule of $M$ that is a composition factor of $FT$. Hence, $d(T) \geq d(FT) \geq d(S) \geq r$. The proofs for $FM$ and simple quotients are similar. □

For $1 \leq i \leq j \leq n$, denote by $G_{[i,j]}$ the symmetric group on $[i, j] = \{i, i + 1, \ldots, j\}$. We define similarly $G_{q_{[i,j]}}$ and $\mathcal{H}_{q_{[i,j]}}$ and we put $c_{[i,j]} = \sum_{w \in G_{[i,j]}} q^{-\ell(w)\tau}(T_w)T$.

Lemma 1.3. The $G_{q_{[i,j]}}$-module $c_{[i+1,j]}EF$ has a simple socle and head.

Proof. See [CR] Lemma 3.6, or [Ve] Theorem 5.10. □

Proof of Proposition 1.1. The proof is in several steps.

Step 1. (a) holds when $FS = 0$. Since $[E]$, $[F]$ define an $SL$-action on $[C]$, $[F(n)]E(n)S = rS$ for some $r \in \mathbb{Z}_{>0}$. By adjointness, $\text{Hom}(F^{(n)}E^{(n)}S, S) = \text{Hom}(E^{(n)}S, E^{(n)}S) \neq 0$. So there exists a nonzero homomorphism $F^{(n)}E^{(n)}S \to S$, hence an isomorphism. Then $F^{(n)}E^{(n)}S \cong S$. If $E^{(n)}S$ has at least two composition factors, then by weight consideration, $F^{(n)}E^{(n)}S$ also has at least two composition factors, and thus cannot be simple. So $E^{(n)}S = S'$ must be simple.

Step 2. (a) holds in general. Let $L$ be a simple quotient of $F^{(r)}S$, where $r = \dim(S)$. Note that, by our choice of $r$, $FL = 0$ so, by Step 1, $E^{(n+r)}L = T$ is simple and $E^{(n)}E^{(r)}L = (^{n+r})T$. By adjunction, we have that $\text{Hom}(S, E^{(n)}L) \cong \text{Hom}(F^{(r)}S, L) \neq 0$, so $S$ must be a subobject of $E^{(n)}L$. It follows that $E^{(n)}S$ must be a subobject of $(^{r})T$. So $E^{(n)}S = mT$ for some $m > 0$. Clearly, $m = \dim\text{Hom}(E^{(n)}S, T) = \dim\text{Hom}(S, E^{(n)}T)$. But $ET = 0$, so by Step 1 (with $E$ and $F$ swapped) $\text{soc}(F^{(n)}T)$ is simple. Thus, $m = 1$.

Step 3. (b) holds whenever (a) does. Clearly, (b) holds when $i = n$. But let us observe a bit more. We have $E^{n}S = n!S'$ for some simple module $S'$. Thus, $E^{n}S = S' \otimes R$ for some left $G_{q_{[1,n]}}$-module $R$ in $N_n$. Since $\dim R = n! = \dim K_n$, we must have $R = K_n$.

For $i < n$ we have, using exactness of $E$ and the above paragraph, that $E^{n-i}S \otimes K_{n-i}c_{[1,n]}^1$. The $G_{q_{[n-i]}}$-module $K_{n-i}c_{[1,n]}^1$ has a simple head and socle, (Lemma 1.3), so the same is true for $S'' \otimes K_{n-i}c_{[1,n]}^1$ (as a $G_{q_{[n-i]}}$-module in $C$). It follows that $E^{n-i}S \otimes K_{n-i}c_{[1,n]}$ is indecomposable as a $G_{q_{[n-i]}}$-module in $C$. Now, if $S'$ is a nonzero summand of $E^{(n)}S$, then $E^{n-i}S' \neq 0$ (Lemma 1.2). So $\text{soc}(E^{(n)}S)$ has no more than one summand and hence must be simple. We have $\text{soc}(E^{(n)}S) \cong S' \otimes R$ for some $G_{q_{[n-i]}}$-module $R$ in $N_i$. Since $\dim R = i!$, it follows that $R \cong K_i$, $\text{soc}(E^{(n)}S) = S'$. The proof for the head being simple is similar. It remains to show that the head and the socle are isomorphic.

Step 4. Estimating the dimension of $\text{End}(E^{(n)}S)$. Since $S' = \text{soc}(E^{(n)}S)$ is simple, the dimension of $\text{Hom}(M, E^{(n)}S)$ is at most the multiplicity of $S'$ in $M$. Taking $M = E^{(n)}S$, we get that the dimension of $\text{End}(E^{(n)}S)$ is at most the multiplicity of $S'$ in $E^{(n)}S$. Since $E^{(n-i)}S' \neq 0$, we have that the dimension of $\text{End}(E^{(n)}S)$ is at most the number of composition factors of $E^{(n-i)}E^{(n)}S$. But $E^{(n-i)}E^{(n)}S = (^n)S''$. Thus, dim(End$(E^{(n)}S)) \leq (^n)$. Since $E^{(n)}S = n!E^{(n)}S$, it follows that dim(End$(E^{(n)}S) \leq (i!^2 (n)) = \dim B_{i,n}$.

Step 5. (c) holds whenever (a) holds. ker$\gamma_n(S) \subseteq n_{q_{[1,n]}}$ since the former is a proper ideal and the latter is a maximal ideal of $H_{q_{[1,n]}}$. For $i < n$, we have that ker$\gamma_i(S) \subseteq H_{q_{[1,i]}}(i) \cap \text{ker} \gamma_n(S) \subseteq H_{q_{[1,i]}}(i) \cap (n_{q_{[1,i]}})$. Then, we have an induced surjective map im$\gamma_i(S) \to B_{i,n}$. By Step 4 (that was done under the assumption that (a) holds) this must be an isomorphism and $\gamma_i(S)$ must be surjective.

Step 6. (d) holds whenever (a) holds. In Step 4 we also get that the multiplicity of $S'$ as a composition factor of $E^{(n)}S$ is $^{(n)}$. If $L$ is a composition factor of $E^{(n)}S$ with $E^{(n-i)}L \neq 0$, then $L \cong S'$. And since the multiplicity of head$(E^{(n)}S)$ in $E^{(n)}S$ is also $^{(n)}$ and head$(E^{(n)}S)$ is not killed by $E^{(n-i)}$, head$(E^{(n)}S) \cong S' \cong \text{soc}(E^{(n)}S)$. Now we also finish the proof of (b) and we are done. □

Take $i = 1$ in the proposition above, we get a map

\[(1) \quad \hat{c} : \text{Irr}C \to \text{Irr}C \cup \{0\}, \quad S \mapsto \text{soc}(ES) = \text{head}(ES),\]
and similarly
\[ \tilde{f} : \text{Irr} \mathcal{C} \to \text{Irr} \mathcal{C} \cup \{0\}, \quad S \mapsto \text{soc}(FS) = \text{head}(FS). \]
Note that if \( ES = 0 \) then \( \hat{c}(S) = \text{soc}(ES) = 0; \) if \( \hat{c}(S) \neq 0 \), we have \( \tilde{f} \hat{c}S = S \).

2. BERENSTEIN-KAZHDAN PERFECT BASES

In this section we introduce the Berenstein-Kazhdan perfect bases. A \( g \)-module, a basis is perfect in the sense that it behaves nicely under the action of Chevalley generators. It equips the \( g \)-module with a crystal structure, which was first defined by Kashiwara using quantum groups. The main reference of this section is \([BK, \text{Section 5}]\).

Let \( I \) be a finite set of indices. Let \( \Lambda \) be a lattice and \( \Lambda' = \Lambda^* \) be its dual lattice, and let \( \{ \alpha_i : i \in I \} \) be a subset of \( \Lambda \) and \( \{ \alpha_i^\vee : j \in I \} \) be a subset of \( \Lambda'^* \). Denote by \( g \) the Kac-Moody algebra associated to the Cartan matrix \( A = (a_{ij})_{i,j \in I} \) with \( a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \), where \( \langle \cdot, \cdot \rangle \) is the evaluation pairing. Also denote by \( e_i, f_i, h_i \in I \) the Chevalley generators of \( g \). We say a \( g \)-module \( V \) is an integrable highest weight module if:

- \( V \) admits a weight decomposition \( V = \bigoplus V_\lambda \) and the weights are bounded above.
- \( e_i \) and \( f_i \) act locally nilpotently for \( i \in I \), i.e., for any \( v \in V \) and any \( i \in I \), there exists an integer \( N \) such that \( e_i^N(v) = 0 \) and \( f_i^N(v) = 0 \).

For a non-zero vector \( v \in V \) and \( i \in I \), denote by \( h_i(v) \) the smallest positive integer \( j \) such that \( e_i^{j+1}(v) = 0 \) and we use the convention \( h_i(0) = -\infty \) for \( v = 0 \). Similarly \( h_i^-(v) = \min \{ j \in \mathbb{Z} : f_i^{j+1}(v) = 0 \} \). Further, denote \( d_i(v) := h_i^+(v) + h_i^-(v) + 1 \) to be the maximal dimension of the irreducible \( sl_2 \)-submodule in \( U(g) v \), where \( g_i \) is the subalgebra of \( g \) generated by \( e_i, f_i \) and \( h_i = [e_i, f_i] \).

For each \( i \in I \) and \( d \geq 0 \), define the subspace
\[ V_{i < d} := \{ v \in V : d_i(v) < d \}. \]
We say that a basis \( B \) of a integrable highest weight \( g \)-module \( V \) is a weight basis if \( B \) is compatible with the weight decomposition, i.e., \( B_\lambda := V_\lambda \cap B \) is a basis of \( V_\lambda \) for any \( \lambda \) being a weight of \( V \).

**Definition 2.1.** We say that a weight basis \( B \) in an integrable highest weight \( g \)-module \( V \) is perfect if for each \( i \in I \) there exist maps \( \hat{e}_i, \hat{f}_i : B \to B \cup \{0\} \) such that \( \hat{e}_i(b) \in B \) if and only if \( e_i(b) \neq 0 \), and in the latter case on has
\[ e_i(b) \in \mathbb{C}^x \cdot \hat{e}_i(b) + V_{i < d_i(b)}; \]
and \( \hat{f}_i(b) \in B \) if and only if \( f_i(b) \neq 0 \), and in the latter case on has
\[ f_i(b) \in \mathbb{C}^x \cdot \hat{f}_i(b) + V_{i < d_i(b)}. \]
We refer to a pair \((V, B)\), where \( V \) is an integrable highest weight \( g \)-module and \( B \) is a perfect basis of \( V \), as a based \( g \)-module.

Denote by \( V^+ \) the space of the highest weight vectors of \( V \):
\[ V^+ = \{ v \in V : e_i(v) = 0, \forall i \in I \}. \]
Denote \( B^+ := B \cap V^+ \). Then we have the following result.

**Proposition 2.2.** For any perfect basis \( B \) for \( V \), the subset \( B^+ \) is a basis for \( V^+ \).

**Proof.** For \( v \in V^+ \), \( e_i(v) = 0 \) for all \( i \in I \). \( B \) is a basis of \( V \), so \( v = \sum_{b \in B} \alpha_b b \) with \( \alpha_b \in \mathbb{C} \). Therefore
\[ e_i(v) = \sum_{b \in B} \alpha_b e_i(b) = \sum_{b \in B, e_i(b) \neq 0} \alpha_b e_i(b) = 0. \]
\( B \) is perfect so by equation (3), if \( e_i(b) \neq 0 \) then for some \( x_b \in V_{i < d_i(b)} \) and \( \beta_b \in \mathbb{C}^x \),
\[ e_i(b) = \beta_b \hat{e}_i(b) + x_b. \]
Hence
\[ \sum_{b \in B, e_i(b) \neq 0} (\alpha_b \beta_b \hat{e}_i(b) + \alpha_b x_b) = 0. \]
Take \( n = \max \{ h_i^+(\hat{e}_i(b)) : b \in B, e_i(b) \neq 0 \} \) and \( B_n := \{ b \in B : \alpha_b \neq 0, h_i^+(\hat{e}_i(b)) = n \} \). Then
\[ e_i^n(e_i(v)) = 0 = \sum_{b \in B_n} \alpha_b \beta_b e_i^n(\hat{e}_i(b)). \]
Note that for any \( b \in B_n \), \( \beta_b \neq 0 \) and \( e_i^n(\hat{e}_i(b)) \neq 0 \). So \( \alpha_b = 0 \) and \( B_n \) is empty. So for any \( b \in B \) such that \( \alpha_b \neq 0 \), \( h_i^+(\hat{e}_i(b)) = 0 \). So \( h_i^+(\hat{e}_i(b)) = 0 \) and \( b \in B^+ \). \( \square \)
3. Perfect basis in $[\mathcal{C}]$

Recall from [Si] Section 2.5, if given $q \neq 1$ being a primitive $l$th-root of unity in $\mathbb{F}$ and $q = (q_0, \cdots, q_{l-1}) \in \mathbb{F}^l$ with $q_i = q^{k_i}$ for $k_i \in \mathbb{Z}/l\mathbb{Z},$ we can construct an $\widehat{\mathfrak{sl}_l}$-categorification on $\mathcal{C} = \bigoplus_{n \geq 0} H_n - \text{mod}$, where $H_n = H_{n,q}(q(n))$ denotes the cyclotomic Hecke algebra, which is the quotient of the affine Hecke algebra $H^\text{aff}_n(n)$ by the extra relation $(X_1 - q_0) \cdots (X_1 - q_{l-1}) = 0$ (which is also called a cyclotomic polynomial). The categorification data is given as follows:

- The biadjoint endofunctors $E = \bigoplus \text{Res}_{n+1}^l$ and $F = \bigoplus \text{Ind}_{n+1}^l$, with the decompositions $E = \bigoplus_{i=0}^{l-1} E_i$ and $F = \bigoplus F_i$, where $E_i$ is the $i$-Restriction and $F_i$ is the $i$-Induction, defined in [Si] Section 2.4.
- $L = \bigoplus L_n \in \text{End}(E)$ with $L_n$ denoting the $n$-th Jucys-Murphy element in $H_n$.
- $T = \bigoplus T_{n-1} \in \text{End}(E^2)$ with $T_{n-1} \in H_n$ being a particular generator of the cyclotomic Hecke algebra.

For $i = 0, \cdots, l - 1$, $[E_i]$ and $[F_i]$ define a $\mathfrak{sl}_2$-action on $[\mathcal{C}] = K_0(\mathcal{C}) \otimes \mathbb{C}$. It is mentioned in [Si] Proposition 3.4 that we have the weight decomposition $\mathcal{C} = \bigoplus \mathcal{C}_\lambda$, where $\mathcal{C}_\lambda$ is the full subcategory of $\mathcal{C}$ consisting of objects whose class is in the weight space $[\mathcal{C}]_\lambda$.

The reason why we are interested in crystals is that the categorical $\widehat{\mathfrak{sl}_l}$ action on $\mathcal{C}$ gives rise to a canonical crystal structure on the set $\text{Irr}\mathcal{C}$ of simple objects in $\mathcal{C}$. In this section, we are going to construct a perfect basis for the $\widehat{\mathfrak{sl}_l}$-module $[\mathcal{C}]$ using results in Proposition [1.1] and deduce that $[\mathcal{C}]$ is an irreducible $\widehat{\mathfrak{sl}_l}$-module.

Denote $V = [\mathcal{C}]$. According to the weight decomposition, $V$ is an integrable highest weight $\mathfrak{g}$-module. Take the basis $B$ of $V = [\mathcal{C}]$ consisting of classes of all simple objects. Similarly to Equation (1) and (2), we can define maps $\tilde{e}_i, \tilde{f}_i : \text{Irr}\mathcal{C} \rightarrow \text{Irr}\mathcal{C} \sqcup \{0\}$ for $i \in I$. Note that for a simple object $S$ in $\mathcal{C}$, $\tilde{e}_i(S) = 0$ if and only if $\text{soc } E_i(S) = 0$, iff and only if $E_iS = 0$, i.e., $e_i[S] = 0$. Together with Proposition [1.1], we see that $\tilde{e}_i, \tilde{f}_i$ are maps satisfying conditions (3) and (4), so $B = \text{Irr}\mathcal{C}$ is a perfect basis of $V$ and $(V, B)$ is a based $\mathfrak{g}$-module.

Now consider the basis $B^+$ of the space of highest weight vectors. $[S] \in B^+$ means that $S$ is simple and $\tilde{e}_i([S]) = 0$ for all $i \in I$. Then $e_i[S] = 0$, which means exactly $E_iS = 0$ for all $i \in I$. So $ES = \bigoplus E_iS = 0$, i.e., $\text{res}^n S = 0$ for all $n \geq 0$. The only simple $S$ in $\mathcal{C}$ is a simple $H_0$-module. Since $H_0 = \mathbb{F}$, so $S \simeq \mathbb{F}$ is unique up to isomorphism. $[S]$ is the unique (up to scalar) highest weight vector in $V$. Therefore $V$ is irreducible.

References


[Si] J. Simental, Introduction to type A categorical $\mathfrak{Kac-Moody}$ actions. Notes for this seminar.