HYPERPLANE ARRANGEMENTS
AT THE CROSSROADS OF TOPOLOGY AND COMBINATORICS

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1. Hyperplane arrangements
   - Complement and intersection lattice
   - Cohomology ring
   - Fundamental group

2. Cohomology jump loci
   - Characteristic varieties
   - Resonance varieties
   - The Tangent Cone theorem

3. Jump loci of arrangements
   - Resonance varieties
   - Multinets
   - Characteristic varieties

4. The Milnor fibrations of an arrangement
   - The Milnor fibrations of an arrangement
   - The homology of the Milnor fiber
   - Modular inequalities
   - Torsion in homology
**Hyperplane arrangements**

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear subspaces in $\mathbb{C}^\ell$.

- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.

- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. 
Example (The Boolean arrangement)
- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

Example (The braid arrangement)
- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \ldots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: configuration space of $n$ ordered points in $\mathbb{C}$ (a classifying space for $P_n$, the pure braid group on $n$ strings).
**Figure**: A planar slice of the braid arrangement $A_4$
We may assume that \( \mathcal{A} \) is essential, i.e., \( \bigcap_{H \in \mathcal{A}} H = \{0\} \).

Fix an ordering \( \mathcal{A} = \{H_1, \ldots, H_n\} \), and choose linear forms \( f_i : \mathbb{C}^\ell \to \mathbb{C} \) with \( \ker(f_i) = H_i \). Define an injective linear map

\[
\iota : \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \ldots, f_n(z)).
\]

This map restricts to an inclusion \( \iota : M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n) \). Hence, \( M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n \) is a Stein manifold.

Therefore, \( M = M(\mathcal{A}) \) has the homotopy type of a connected, finite cell complex of dimension \( \ell \).

In fact, \( M \) has a minimal cell structure (Dimca–Papadima, Randell, Salvetti, Adiprasito, . . . ). Consequently, \( H_*(M, \mathbb{Z}) \) is torsion-free.
**Cohomology ring**

- The Betti numbers \( b_q(M) := \text{rank } H_q(M, \mathbb{Z}) \) are given by
  \[ \sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(A)} \mu(X) (-t)^{\text{rank}(X)}, \]
  where \( \mu : L(A) \to \mathbb{Z} \) is the Möbius function, defined recursively by
  \( \mu(\mathbb{C}^\ell) = 1 \) and \( \mu(X) = -\sum_{Y \supseteq X} \mu(Y) \).

- Let \( E = \wedge(A) \) be the \( \mathbb{Z} \)-exterior algebra on degree 1 classes \( e_H \) dual to the meridians around the hyperplanes \( H \in A \).

- Let \( \partial : E^\bullet \to E^{\bullet-1} \) be the differential given by \( \partial(e_H) = 1 \), and set
  \( e_B = \prod_{H \in B} e_H \) for each \( B \subset A \).

- The cohomology ring \( H^*(M(A), \mathbb{Z}) \) is isomorphic to the Orlik–Solomon algebra \( A(A) = E/I \), where
  \[ I = \text{ideal } \left\langle \partial e_B \mid \text{codim } \bigcap_{H \in B} H < |B| \right\rangle. \]
Given a generic projection of a generic slice of $\mathcal{A}$ in $\mathbb{C}^2$, the fundamental group $\pi = \pi_1(M(\mathcal{A}))$ can be computed from the resulting braid monodromy $\alpha = (\alpha_1, \ldots, \alpha_s)$, where $\alpha_r \in P_n$.

- $\pi$ has a (minimal) finite presentation with
  - Meridional generators $x_1, \ldots, x_n$, where $n = |\mathcal{A}|$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where each $\alpha_j$ acts on $F_n$ via the Artin representation.

Let $\pi/\gamma_k(\pi)$ be the $(k - 1)^{th}$ nilpotent quotient of $\pi$. Then:

- $\pi_{ab} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
- $\pi/\gamma_3$ is determined by $A^{\leq 2}(\mathcal{A})$, and thus by $L^{\leq 2}(\mathcal{A})$.
- $\pi/\gamma_4$ (and thus, $\pi$) is not determined by $L(\mathcal{A})$. (Rybnikov).
Let $X$ be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

Let $k$ be an algebraically closed field, and let $\text{Hom}(\pi, k^*)$ be the affine algebraic group of $k$-valued, multiplicative characters on $\pi$.

The *characteristic varieties* of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$V^q_s(X, k) = \{ \rho \in \text{Hom}(\pi, k^*) \mid \dim_k \, H_q(X, k\rho) \geq s \}.$$ 

Here, $k\rho$ is the local system defined by $\rho$, i.e., $k$ viewed as a $k\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, k\rho) = H_i(C_*(\tilde{X}, k) \otimes_{k\pi} k\rho)$.

These loci are Zariski closed subsets of the character group.

The sets $V^q_s(X, k)$ depend only on $\pi/\pi''$. 
**Example (Circle)**

We have $\widetilde{S}^1 = \mathbb{R}$. Identify $\pi_1(S^1, \ast) = \mathbb{Z} = \langle t \rangle$ and $k\mathbb{Z} = k[t^\pm 1]$. Then:

$$C_\ast(\widetilde{S}^1, k) : 0 \rightarrow k[t^\pm 1] \xrightarrow{t^{-1}} k[t^\pm 1] \rightarrow 0.$$ 

For $\rho \in \text{Hom}(\mathbb{Z}, k^*) = k^*$, we get

$$C_\ast(\widetilde{S}^1, k) \otimes_{k\mathbb{Z}} k\rho : 0 \rightarrow k \xrightarrow{\rho^{-1}} k \rightarrow 0,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, k) = H_1(S^1, k) = k$.

Hence: $V^0_1(S^1, k) = V^1_1(S^1, k) = \{1\}$ and $V^i_s(S^1, k) = \emptyset$, otherwise.

**Example (Punctured complex line)**

Identify $\pi_1(\mathbb{C}\setminus\{n \text{ points}\}) = F_n$, and $\widehat{F}_n = (k^*)^n$. Then:

$$V^1_s(\mathbb{C}\setminus\{n \text{ points}\}, k) = \begin{cases} (k^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$
Resonance varieties

Let $A = H^*(X, \mathbb{k})$, where $\text{char} \, \mathbb{k} \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.

We thus get a cochain complex

$$(A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \rightarrow \cdots.$$  

The resonance varieties of $X$ are the jump loci for the cohomology of this complex

$$R^q_s(X, \mathbb{k}) = \{ a \in A^1 | \dim_{\mathbb{k}} H^q(A, \cdot a) \geq s \}.$$  

E.g., $R^1_1(X, \mathbb{k}) = \{ a \in A^1 | \exists b \in A^1, \ b \neq \lambda a, \ ab = 0 \}.$

These loci are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{k})$.

Example

$R^1_1(T^n, \mathbb{k}) = \{0\}$, for all $n > 0$.

$R^1_1(\mathbb{C}\{n \text{ points}\}, \mathbb{k}) = \mathbb{k}^n$, for all $n > 1$. 
The Tangent Cone Theorem

Given a subvariety $W \subset (\mathbb{C}^*)^n$, let
$$\tau_1(W) = \{ z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C} \}.$$  

(Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq TC_1(W)$.

(Libgober 2002/DPS 2009)
$$\tau_1(\mathcal{V}_s^i(X)) \subseteq TC_1(\mathcal{V}_s^i(X)) \subseteq R_s^i(X).$$

(DPS 2009/DP 2014): Suppose $X$ is a $k$-formal space. Then, for each $i \leq k$ and $s > 0$,
$$\tau_1(\mathcal{V}_s^i(X)) = TC_1(\mathcal{V}_s^i(X)) = R_s^i(X).$$

Consequently, $R_s^i(X, \mathbb{C})$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$. 
Work of Arapura, Falk, D.Cohen–A.S., Libgober, and Yuzvinsky, completely describes the varieties $\mathcal{R}_s(\mathcal{A}) := \mathcal{R}_s^1(M(\mathcal{A}), \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{\vert \mathcal{A} \vert}$.

- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s + 1$.

- (Falk–Yuzvinsky 2007) Each $k$-multinet on a sub-arrangement $\mathcal{B} \subset \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k - 1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.
Multinets

- To compute $R_1(\mathcal{A})$, we may assume $\mathcal{A}$ is an arrangement in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.

- $L_1(\mathcal{A}) \leftrightarrow$ lines of $\bar{\mathcal{A}}$, $L_2(\mathcal{A}) \leftrightarrow$ intersection points of $\bar{\mathcal{A}}$.

- A flat $X \in L_2(\mathcal{A})$ has multiplicity $q$ if the point $\bar{X}$ has exactly $q$ lines from $\bar{\mathcal{A}}$ passing through it.

- A $(k, d)$-multinet on $\mathcal{A}$ is a partition into $k \geq 3$ subsets, $\mathcal{A}_1, \ldots, \mathcal{A}_k$, together with an assignment of multiplicities, $m: \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, such that (basically):
  
  $\exists \ d \in \mathbb{N}$ such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.

  $\text{2}$ If $H$ and $H'$ are in different classes, then $H \cap H' \in \mathcal{X}$.

  $\text{3}$ $\forall X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supseteq X} m_H$ is independent of $\alpha$.

- The multinet is reduced if $m_H = 1$, for all $H \in \mathcal{A}$.

- A net is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$. 
**Example (Braid arrangement $\mathcal{A}_4$)**

$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3, 2)$-net:

- $L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}$,
- $L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\}$,
- $L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}$,
- $L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\}$,
- $L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}$. 
Let $\text{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.

The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1(M(\mathcal{A}), \mathbb{C})$ lies in the substorus $\{ t \in (\mathbb{C}^*)^n | t_1 \cdots t_n = 1 \}$.

$\mathcal{V}_1(\mathcal{A})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.

If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.

All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$. 
Suppose there is a multinet $\mathcal{M}$ on $\mathcal{A}$, and there is a hyperplane $H$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

Then $\mathcal{V}_1(\mathcal{A} \setminus \{H\})$ has a component which is a 1-dimensional subtorus, translated by a character of order $m_H$.

**Example (The deleted $B_3$ arrangement)**

The $B_3$ arrangement supports a $(3, 4)$-multinet; $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 quadruple points ($n_X = 2$). So pick $H$ with $m_H = 2$ to get a translated torus in $\mathcal{V}_1(B_3 \setminus \{H\})$. 
The Milnor fibration(s) of an arrangement

- Let $\mathcal{A}$ be a (central) hyperplane arrangement in $\mathbb{C}^\ell$.

- For each $H \in \mathcal{A}$, let $f_H : \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel $H$.

- For each choice of multiplicities $m = (m_H)_{H \in \mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.

- The map $Q_m : \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m : M(\mathcal{A}) \to \mathbb{C}^*$.

- This is the projection of a smooth, locally trivial bundle, known as the Milnor fibration of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \to M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$
The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the 
Milnor fiber of the multi-arrangement.

$F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $\ell - 1$.

The (geometric) monodromy is the diffeomorphism

$$h: F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i/N} z.$$ 

If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$. 

**Example**

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = m$-roots of 1.

**Example**

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and $h$ is an automorphism of order 3:

More generally, if $\mathcal{A}$ is a pencil of $n$ lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\left(\frac{n-1}{2}\right)$, with $n$ punctures.
Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential arrangement. The inclusion $\iota: M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{align*}
F_m(\mathcal{A}) \to M(\mathcal{A}) &\xrightarrow{Q_m(\mathcal{A})} \mathbb{C}^* \\
F_m(\mathcal{B}_n) \to M(\mathcal{B}_n) &\xrightarrow{Q_m(\mathcal{B}_n)} \mathbb{C}^*
\end{align*}$$

Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$
The homology of the Milnor fiber

Let \((\mathcal{A}, m)\) be a multi-arrangement with \(\gcd(m) = 1\). Set \(N = \sum_{H \in \mathcal{A}} m_H\).

The Milnor fiber \(F_m(\mathcal{A})\) is a regular \(\mathbb{Z}_N\)-cover of the projectivized complement, \(U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))\), defined by the homomorphism

\[
\delta_m : \pi_1(U(\mathcal{A})) \rightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \mod N
\]

Let \(\widehat{\delta}_m : \text{Hom}(\mathbb{Z}_N, k^*) \rightarrow \text{Hom}(\pi_1(U(\mathcal{A})), k^*)\) be the induced map between character groups.

If \(\text{char}(k) \nmid N\), the dimension of \(H_q(F_m(\mathcal{A}), k)\) may be computed by summing up the number of intersection points of \(\text{im}(\widehat{\delta}_m)\) with the varieties \(\mathcal{V}_s^q(U(\mathcal{A}), k)\), for all \(s \geq 1\).
We now consider the simplest non-trivial case: that of an arrangement \( A \) of \( n \) planes in \( \mathbb{C}^3 \), and its Milnor fiber, \( F(A) \).

Let \( \Delta_A(t) = \det(t \cdot \text{id} - h_*) \) be the characteristic polynomial of the algebraic monodromy, \( h_* : H_1(F(A), \mathbb{C}) \to H_1(F(A), \mathbb{C}) \).

Since \( h^n_\bullet = \text{id} \), we may write

\[
\Delta_A(t) = \prod_{d|n} \Phi_d(t)^{e_d(A)},
\]

where \( \Phi_d(t) \) is the \( d \)-th cyclotomic polynomial, and \( e_d(A) \in \mathbb{Z}_{\geq 0} \).

**Problem**

- Is the polynomial \( \Delta_A \) (or, equivalently, the exponents \( e_d(A) \)) determined by the intersection lattice \( L(A) \)?
- In particular, is the first Betti number \( b_1(F(A)) = \deg(\Delta_A) \) combinatorially determined?
By a transfer argument, $e_1(\mathcal{A}) = n - 1$.

Not all divisors of $n$ appear in $(*)$. E.g., if $d$ does not divide at least one of the multiplicities of the intersection points, then $e_d(\mathcal{A}) = 0$.

In particular, if $\mathcal{A}$ has only points of multiplicity 2 and 3, then $\Delta_{\mathcal{A}}(t) = (t - 1)^{m-1}(t^2 + t + 1)^{e_3}$.

If multiplicity 4 appears, then also get factor of $(t + 1)^{e_2} \cdot (t^2 + 1)^{e_4}$.

**Example**

Let $\mathcal{A} = \mathcal{A}_4$ be the braid arrangement. Then $\mathcal{V}_1(\mathcal{A})$ has a single ‘essential’ component,

$$T = \{ t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_1 t_6^{-1} = t_2 t_5^{-1} = t_3 t_4^{-1} = 1 \}.$$  

Clearly, $\delta^2 \in T$, yet $\delta \notin T$. Hence, $\Delta_{\mathcal{A}}(t) = (t - 1)^5(t^2 + t + 1)$. 
MODULAR INEQUALITIES

- Let $\sigma = \sum_{H \in A} e_H \in A^1$ be the “diagonal” vector.
- Assume $k$ has characteristic $p > 0$, and define
  \[ \beta_p(A) = \dim_k H^1(A, \sigma). \]
  That is, $\beta_p(A) = \max\{s \mid \sigma \in R^1_s(A, k)\}$.


$e_{ps}(A) \leq \beta_p(A)$, for all $s \geq 1$.

**Theorem**

1. Suppose $A$ admits a $k$-net. Then $\beta_p(A) = 0$ if $p \nmid k$ and $\beta_p(A) \geq k - 2$, otherwise.
2. If $A$ admits a reduced $k$-multinet, then $e_k(A) \geq k - 2$. 

**THEOREM**
**Theorem (Papadima–S. 2014)**

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced $3$-multinet iff $\mathcal{A}$ admits a $3$-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leq 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.

**Corollary (PS)**

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity $2$ or $3$. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

**Theorem (PS)**

Suppose $\mathcal{A}$ supports a $4$-net and $\beta_2(\mathcal{A}) \leq 2$. Then

$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2$. 
**Conjecture (PS)**

Let $\mathcal{A}$ be an arrangement which is not a pencil. Then $e_{ps}(\mathcal{A}) = 0$ for all primes $p$ and integers $s \geq 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors $d$ of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t - 1)^{|\mathcal{A}| - 1}((t + 1)(t^2 + 1))^{\beta_2(\mathcal{A})}(t^2 + t + 1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements:

- Complex reflection arrangements (Măcinic–Papadima–Popescu).
- Certain types of real arrangements (Yoshinaga, Bailet, Torielli).
**Torsion in homology**

**Theorem (Cohen–Denham–S. 2003)**

For every prime $p \geq 2$, there is a multi-arrangement $(A, m)$ such that $H_1(F_m(A), \mathbb{Z})$ has non-zero $p$-torsion.

Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(A) = x^2y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)$$

Then $H_1(F_m(A), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 
We now can generalize and reinterpret these examples, as follows.

A **pointed multinet** on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subseteq H$.

**Theorem (Denham–S. 2014)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1(M(\mathcal{A}'), k)$ varies with $\text{char}(k)$. 
To produce $p$-torsion in the homology of the usual Milnor fiber, we use a “polarization” construction:

$\langle \mathcal{A}, m \rangle \rightsquigarrow \mathcal{A} \| m$, an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\text{rank } \mathcal{A} + |\{H \in \mathcal{A}: m_H \geq 2\}|$.

**Theorem (DS)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$.

There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has $p$-torsion, where $\mathcal{B} = \mathcal{A}' \| m'$ and $q = 1 + |\{K \in \mathcal{A}' : m'_K \geq 3\}|$. 
**Corollary (DS)**

For every prime $p \geq 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion, for some $q > 1$.

Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1w_2w_3w_4w_5(x^2 - w_1^2)(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1).$$

$$((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)((x - z)^2 - w_4^2)((x - z)^2 - 2w_4^2) \cdot ((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).
REFERENCES

