Abelian Galois covers and rank one local systems

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   - The Dwyer–Fried sets

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Galois covers

Sample questions:

1. Given a (finite) CW-complex $X$, how to parametrize the Galois covers of $X$ with fixed deck-transformation group $A$?

2. Given an infinite Galois $A$-cover, $Y \rightarrow X$, are the Betti numbers of $Y$ finite?
   - If so, how to compute the Betti numbers of $Y$?
   - Furthermore, do the Galois covers of $Y$ have finite Betti numbers?

3. Do the Galois $A$-covers that have finite Betti numbers form an open subspace of the parameter space?

4. Given a finite Galois $A$-cover, $Y \rightarrow X$, how to compute the Betti numbers of $Y$?
Let $X$ be a connected CW-complex with finite 1-skeleton. We may assume $X$ has a single 0-cell, call it $x_0$. Set $G = \pi_1(X, x_0)$.

Any epimorphism $\nu: G \twoheadrightarrow A$ gives rise to a (connected) Galois cover, $X^\nu \rightarrow X$, with group of deck transformations $A$.

Moreover, if $\alpha \in \text{Aut}(A)$, then $X^{\alpha \circ \nu} \cong X^\nu$ ($A$-equivariant homeo).

Conversely, if $p: (Y, y_0) \rightarrow (X, x_0)$ is a Galois $A$-cover, we get a short exact sequence

$$1 \rightarrow \pi_1(Y, y_0) \xrightarrow{p_\#} \pi_1(X, x_0) \xrightarrow{\nu} A \rightarrow 1,$$

and an $A$-equivariant homeomorphism $Y \cong X^\nu$.

Thus, the set of Galois $A$-covers of $X$ can be identified with

$$\text{Epi}(G, A)/\text{Aut}(A).$$
Now assume \( A \) is a (finitely generated) Abelian group. Then \( \text{Hom}(G, A) \leftrightarrow \text{Hom}(H, A) \), where \( H = G_{\text{ab}} \).

**Proposition (A.S.–Yang–Zhao)**

*There is a bijection*

\[
\text{Epi}(H, A)/\text{Aut}(A) \leftrightarrow \text{GL}_n(\mathbb{Z}) \times P \Gamma
\]

*where \( n = \text{rank } H, \ r = \text{rank } A, \) and*

- \( P \) is a parabolic subgroup of \( \text{GL}_n(\mathbb{Z}) \);
- \( \text{GL}_n(\mathbb{Z})/P = \text{Gr}_{n-r}(\mathbb{Z}^n) \);
- \( \Gamma = \text{Epi}(\mathbb{Z}^{n-r} \oplus \text{Tors}(H), \text{Tors}(A))/\text{Aut}(\text{Tors}(A)) \) — a finite set;
- \( \text{GL}_n(\mathbb{Z}) \times P \Gamma \) is the twisted product under the diagonal \( P \)-action.*
Simplest situation is when $A = \mathbb{Z}^r$.

All Galois $\mathbb{Z}^r$-covers of $X$ arise as pull-backs of the universal cover of the $r$-torus:

\[
\begin{align*}
X^\nu & \rightarrow \mathbb{R}^r \\
\downarrow & \\
X & \rightarrow T^r,
\end{align*}
\]

where $f_\# : \pi_1(X) \rightarrow \pi_1(T^r)$ realizes the epimorphism $\nu : G \rightarrow \mathbb{Z}^r$.

Hence:

\[
\{\text{Galois } \mathbb{Z}^r\text{-covers of } X\} \leftrightarrow \{r\text{-planes in } H^1(X, \mathbb{Q})\}
\]

\[
X^\nu \rightarrow X \leftrightarrow P_\nu
\]

where $P_\nu := \text{im}(\nu^* : H^1(\mathbb{Z}^r, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q}))$.

Thus:

\[
\text{Epi}(H, \mathbb{Z}^r) / \text{Aut}(\mathbb{Z}^r) \cong \text{Gr}_{n-r}(\mathbb{Z}^n) \cong \text{Gr}_r(\mathbb{Q}^n).
\]
The Dwyer–Fried sets

Moving about the parameter space for $A$-covers, and recording how the Betti numbers of those covers vary leads to:

**Definition**

The *Dwyer–Fried invariants* of $X$ are the subsets

$$\Omega^i_A(X) = \{ [\nu] \in \text{Epi}(G, A)/\text{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leq i \}.$$  

where $X^\nu \to X$ is the cover corresponding to $\nu : G \to A$.

In particular, when $A = \mathbb{Z}^r$,

$$\Omega^i_r(X) = \{ P_{\nu} \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i \},$$

with the convention that $\Omega^i_r(X) = \emptyset$ if $r > n = b_1(X)$. For a fixed $r > 0$, get filtration

$$\text{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots.$$
The $\Omega$-sets are homotopy-type invariants: If $X \simeq Y$, then, for each $r > 0$, there is an isomorphism $Gr_r(H^1(Y, \mathbb{Q})) \cong Gr_r(H^1(X, \mathbb{Q}))$ sending each subset $\Omega_r^i(Y)$ bijectively onto $\Omega_r^i(X)$.

Thus, we may extend the definition of the $\Omega$-sets from spaces to groups: $\Omega_r^i(G) = \Omega_r^i(K(G, 1))$, and similarly for $\Omega_r^i_A(X)$.

**Example**

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^{ab} \simeq \bigvee_{j \in \mathbb{Z}} S^k_j$. Thus,

$$\Omega^i_1(X) = \begin{cases} \{\text{pt}\} & \text{for } i < k, \\ \emptyset & \text{for } i \geq k. \end{cases}$$
Comparison diagram

- There is a commutative diagram,

\[
\Omega_i^A(X) \hookrightarrow \text{Epi}(G, A)/\text{Aut } A \cong \text{GL}_n(\mathbb{Z}) \times \mathbb{P} \Gamma
\]

\[
\Omega_i^r(X) \hookrightarrow \text{Gr}_r(\mathbb{Q}^n)
\]

- If \( \Omega_i^r(X) = \emptyset \), then \( \Omega_i^A(X) = \emptyset \).

- The above is a pull-back diagram if and only if:

  If \( X^\nu \) is a \( \mathbb{Z}^r \)-cover with finite Betti numbers up to degree \( i \), then any regular \( \text{Tors}(A) \)-cover of \( X^\nu \) has the same finiteness property.
Example

Let $X = S^1 \vee \mathbb{RP}^2$. Then $G = \mathbb{Z} \ast \mathbb{Z}_2$, $G_{ab} = \mathbb{Z} \oplus \mathbb{Z}_2$, $G_{fab} = \mathbb{Z}$, and

$$X_{fab} \cong \bigvee_{j \in \mathbb{Z}} \mathbb{RP}_j^2, \quad X_{ab} \cong \bigvee_{j \in \mathbb{Z}} S_j^1 \vee \bigvee_{j \in \mathbb{Z}} S_j^2.$$ 

Thus, $b_1(X_{fab}) = 0$, yet $b_1(X_{ab}) = \infty$.

Hence, $\Omega_1^1(X) \neq \emptyset$, but $\Omega_{\mathbb{Z} \oplus \mathbb{Z}_2}^1(X) = \emptyset$. 
Characteristic varieties

- Group of complex-valued characters of $G$:

$$\hat{G} = \text{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$$

- Let $G_{ab} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of $G$. The map $\text{ab}: G \to G_{ab}$ induces an isomorphism $\hat{G}_{ab} \cong \hat{G}$.

- $\hat{G}^0 = (\mathbb{C}^\times)^n$, an algebraic torus of dimension $n = \text{rank } G_{ab}$.

- $\hat{G} = \bigsqcup_{\text{Tors}(G_{ab})} (\mathbb{C}^\times)^n$.

- $\hat{G}$ parametrizes rank 1 local systems on $X$:

$$\rho: G \to \mathbb{C}^\times \xrightarrow{\sim} \mathbb{C}_\rho$$

the complex vector space $\mathbb{C}$, viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$. 
The homology groups of $X$ with coefficients in $\mathbb{C}_\rho$ are defined as

$$H_\bullet(X, \mathbb{C}_\rho) = H_\bullet(\mathbb{C}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\tilde{X}, \mathbb{Z})).$$

where $C_\bullet(\tilde{X}, \mathbb{Z})$ is the $\mathbb{Z}G$-equivariant cellular chain complex of the universal cover of $X$.

**Definition**

The *characteristic varieties* of $X$ are the sets

$$\mathcal{V}_i(X) = \{ \rho \in \hat{G} \mid H_j(X, \mathbb{C}_\rho) \neq 0, \text{ for some } j \leq i \}.$$  

- Get filtration $\{1\} = \mathcal{V}_0(X) \subseteq \mathcal{V}_1(X) \subseteq \cdots \subseteq \hat{G}$.
- If $X$ has finite $k$-skeleton, then $\mathcal{V}_i(X)$ is a Zariski closed subset of the algebraic group $\hat{G}$, for each $i \leq k$.
- The varieties $\mathcal{V}_i(X)$ are homotopy-type invariants of $X$. 
The characteristic varieties may be reinterpreted as the support varieties for the Alexander invariants of $X$.

- Let $X^{ab} \to X$ be the maximal abelian cover. View $H_*(X^{ab}, \mathbb{C})$ as a module over $\mathbb{C}[G_{ab}]$. Then

$$V^i(X) = V\left(\text{ann} \left( \bigoplus_{j \leq i} H_j(X^{ab}, \mathbb{C}) \right)\right).$$

- Let $X^{fab} \to X$ be the max free abelian cover. View $H_*(X^{fab}, \mathbb{C})$ as a module over $\mathbb{C}[G_{fab}] \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $n = b_1(G)$. Then

$$W^i(X) := V^i(X) \cap \hat{G}^0 = V\left(\text{ann} \left( \bigoplus_{j \leq i} H_j(X^{fab}, \mathbb{C}) \right)\right).$$

**Example**

Let $L = (L_1, \ldots, L_n)$ be a link in $S^3$, with complement $X = S^3 \setminus \bigcup_{i=1}^n L_i$ and Alexander polynomial $\Delta_L = \Delta_L(t_1, \ldots, t_n)$. Then

$$V^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\}.$$
The characteristic varieties

\[ V^j(X, k) = \{ \rho \in \text{Hom}(\pi_1(X), k^\times) \mid \dim_k H_i(X, k\rho) \geq j \} \]

can be used to compute the homology of finite abelian Galois covers (work of A. Libgober, E. Hironaka, P. Sarnak–S. Adams, M. Sakuma, D. Matei–A. S. from the 1990s). E.g.:

**Theorem (Matei–A.S. 2002)**

Let \( \nu : \pi_1(X) \to \mathbb{Z}_n \). Suppose \( \bar{k} = k \) and \( \text{char } k \nmid n \), so that \( \mathbb{Z}_n \subset k^\times \). Then:

\[ \dim_k H_1(X^\nu, k) = \dim_k H_1(X, k) + \sum_{1 \neq k \mid n} \varphi(k) \cdot \text{depth}_k(\nu^{n/k}), \]

where \( \text{depth}_k(\rho) = \max\{j \mid \rho \in V^j(X, k)\} \).
Computing the $\Omega$-invariants

**Theorem (Dwyer–Fried 1987, Papadima–S. 2010)**

Let $X$ be a connected CW-complex with finite $k$-skeleton. For an epimorphism $\nu: \pi_1(X) \to \mathbb{Z}^r$, the following are equivalent:

1. The vector space $\bigoplus_{i=0}^{k} H_i(X^{\nu}, \mathbb{C})$ is finite-dimensional.

2. The algebraic torus $T_{\nu} = \text{im} \left( \hat{\nu}: \mathbb{Z}^r \hookrightarrow \widehat{\pi_1(X)} \right)$ intersects the variety $\mathcal{W}^k(X)$ in only finitely many points.

Let $\exp: H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^\times)$ be the coefficient homomorphism induced by the homomorphism $\mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$.

Under the isomorphism $H^1(X, \mathbb{C}^\times) \cong \widehat{\pi_1(X)}$, we have

$$\exp(P_{\nu} \otimes \mathbb{C}) = T_{\nu}.$$
Thus, we may reinterpret the $\Omega$-invariants, as follows:

**Corollary**

$$\Omega_r^i(X) = \{ P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim (\exp(P \otimes \mathbb{C}) \cap W^i(X)) = 0 \}.$$

More generally, for any abelian group $A$:

**Theorem ([SYZ])**

$$\Omega_A^i(X) = \{ [\nu] \in \text{Epi}(H, A)/\text{Aut}(A) \mid \text{im}(\hat{\nu}) \cap V^i(X) \text{ is finite} \}.$$
Characteristic subspace arrangements

Set \( n = b_1(X) \), and identify \( H^1(X, \mathbb{C}) = \mathbb{C}^n \) and \( H^1(X, \mathbb{C}^\times)^0 = (\mathbb{C}^\times)^n \).

Given a Zariski closed subset \( W \subset (\mathbb{C}^\times)^n \), define the *exponential tangent cone* at 1 to \( W \) as

\[
\tau_1(W) = \{ z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C} \}.
\]

**Lemma (Dimca–Papadima–A.S. 2009)**

\( \tau_1(W) \) is a finite union of rationally defined linear subspaces of \( \mathbb{C}^n \).

The *i-th characteristic arrangement* of \( X \), is the subspace arrangement \( \mathcal{C}_i(X) \) in \( H^1(X, \mathbb{Q}) \) defined as:

\[
\tau_1(\mathcal{W}^i(X)) = \bigcup_{L \in \mathcal{C}_i(X)} L \otimes \mathbb{C}.
\]
Theorem

\[ \Omega^i_r(X) \subseteq \left( \bigcup_{L \in \mathcal{C}_i(X)} \{ P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \mid P \cap L \neq \{0\} \} \right)^c. \]

Proof.

Fix an \( r \)-plane \( P \in \text{Gr}_r(H^1(X, \mathbb{Q})) \), and let \( T = \exp(P \otimes \mathbb{C}) \). Then:

\[
P \in \Omega^i_r(X) \iff T \cap \mathcal{W}^i(X) \text{ is finite}
\]

\[
\implies \tau_1(T \cap \mathcal{W}^i(X)) = \{0\}
\]

\[
\iff (P \otimes \mathbb{C}) \cap \tau_1(\mathcal{W}^i(X)) = \{0\}
\]

\[
\iff P \cap L = \{0\}, \text{ for each } L \in \mathcal{C}_i(X),
\]
For “straight” spaces, the inclusion holds as an equality.

If $r = 1$, the inclusion always holds as an equality.

In general, though, the inclusion is strict. E.g., there exist finitely presented groups $G$ for which $\Omega_2^1(G)$ is not open.

**Example**

Let $G = \langle x_1, x_2, x_3 \mid [x_1^2, x_2], [x_1, x_3], x_1[x_2, x_3]x_1^{-1}[x_2, x_3] \rangle$. Then $G_{ab} = \mathbb{Z}^3$, and

$$V^1(G) = \{1\} \cup \{ t \in (\mathbb{C}^\times)^3 \mid t_1 = -1 \}.$$

Let $T = (\mathbb{C}^\times)^2$ be an algebraic 2-torus in $(\mathbb{C}^\times)^3$. Then

$$T \cap V^1(G) = \begin{cases} \{1\} & \text{if } T = \{t_1 = 1\} \\ \mathbb{C}^\times & \text{otherwise} \end{cases}$$

Thus, $\Omega_2^1(G)$ consists of a single point in $\text{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{Q}P^2$, and so it’s not open.
Special Schubert varieties

- Let $V$ be a homogeneous variety in $\mathbb{k}^n$. The set $\sigma_r(V) = \{ P \in \text{Gr}_r(\mathbb{k}^n) \mid P \cap V \neq \{0\} \}$ is Zariski closed.
- If $L \subset \mathbb{k}^n$ is a linear subspace, $\sigma_r(L)$ is the special Schubert variety defined by $L$. If $\text{codim} L = d$, then $\text{codim} \sigma_r(L) = d - r + 1$.

Theorem

\[ \Omega_i^i(X) \subseteq \text{Gr}_r \left( H^1(X, \mathbb{Q}) \right) \setminus \left( \bigcup_{L \in C_i(X)} \sigma_r(L) \right). \]

Thus, each set $\Omega_i^i(X)$ is contained in the complement of a Zariski closed subset of $\text{Gr}_r(H^1(X, \mathbb{Q}))$: the union of the special Schubert varieties corresponding to the subspaces comprising $C_i(X)$.

Corollary

1. If $\text{codim} C_i(X) \geq d$, then $\Omega_i^i(X) = \emptyset$, for all $r \geq d + 1$.
2. If $\tau_1(\mathcal{W}^1(X)) \neq \{0\}$, then $b_1(X^{\text{fab}}) = \infty$. 
Resonance varieties

Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex of finite-dimensional, complex vector spaces,

$$(A, a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots.$$ 

Definition

The resonance varieties of $X$ are the sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^j(A, \cdot a) \neq 0, \text{ for some } j \leq i\}.$$ 

- Get filtration $\mathcal{R}^0(X) \subseteq \mathcal{R}^1(X) \subseteq \cdots \subseteq \mathcal{R}^k(X) \subseteq H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- If $X$ has finite $k$-skeleton, then $\mathcal{R}^i(X)$ is a homogeneous algebraic subvariety of $\mathbb{C}^n$, for each $i \leq k$.
- These varieties are homotopy-type invariants of $X$.
- $\tau_1(\mathcal{W}^i(X)) \subseteq TC_1(\mathcal{W}^i(X)) \subseteq \mathcal{R}^i(X)$. 
Straight spaces

Let $X$ be a connected CW-complex with finite $k$-skeleton.

**Definition**

We say $X$ is $k$-straight if the following conditions hold, for each $i \leq k$:

1. All positive-dimensional components of $W^i(X)$ are algebraic subtori.
2. $TC_1(W^i(X)) = R^i(X)$.

If $X$ is $k$-straight for all $k \geq 1$, we say $X$ is a straight space.

- The $k$-straightness property depends only on the homotopy type of a space.
- Hence, we may declare a group $G$ to be $k$-straight if there is a $K(G, 1)$ which is $k$-straight; in particular, $G$ must be of type $F_k$.
- $X$ is 1-straight if and only if $\pi_1(X)$ is 1-straight.
Theorem

Let $X$ be a $k$-straight space. Then, for all $i \leq k$,

1. $\tau_1(\mathcal{W}^i(X)) = TC_1(\mathcal{W}^i(X)) = R^i(X)$.
2. $R^i(X, \mathbb{Q}) = \bigcup_{L \in \mathcal{C}_i(X)} L$.

In particular, the resonance varieties $R^i(X)$ are unions of rationally defined subspaces.

Example

Let $G$ be the group with generators $x_1, x_2, x_3, x_4$ and relators $r_1 = [x_1, x_2], r_2 = [x_1, x_4][x_2^{-2}, x_3], r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Then

$$R^1(G) = \{ z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0 \},$$

which splits into two linear subspaces defined over $\mathbb{R}$, but not over $\mathbb{Q}$. Thus, $G$ is not 1-straight.
Theorem

Suppose $X$ is $k$-straight. Then, for all $i \leq k$ and $r \geq 1$,

$$
\Omega^i_r(X) = \text{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r(R^i(X, \mathbb{Q})).
$$

In other words, each set $\Omega^i_r(X)$ is the complement of a finite union of special Schubert varieties in the rational Grassmannian; in particular, $\Omega^i_r(X)$ is a Zariski open set.
The structure of the characteristic varieties of smooth, complex projective and quasi-projective varieties (and, more generally, Kähler and quasi-Kähler manifolds) was determined by Beauville, Green–Lazarsfeld, Simpson, Campana, and Arapura in the 1990s.

**Theorem (Arapura 1997)**

Let \( X = \overline{X} \setminus D \), where \( \overline{X} \) is a compact Kähler manifold and \( D \) is a normal-crossings divisor. If either \( D = \emptyset \) or \( b_1(\overline{X}) = 0 \), then each characteristic variety \( \mathcal{V}^i(X) \) is a finite union of unitary translates of algebraic subtori of \( H^1(X, \mathbb{C}^\times) \).

In degree 1, the condition that \( b_1(\overline{X}) = 0 \) if \( D \neq \emptyset \) may be lifted. Furthermore, each positive-dimensional component of \( \mathcal{V}^1(X) \) is of the form \( \rho \cdot T \), with \( T \) an algebraic subtorus, and \( \rho \) a torsion character.
**Theorem (Dimca–Papadima–A.S. 2009)**

Let $X$ be a 1-formal, quasi-Kähler manifold, and let $\{L_\alpha\}$ be the positive-dimensional, irreducible components of $R^1(X)$. Then:

1. Each $L_\alpha$ is a linear subspace of $H^1(X, \mathbb{C})$ of dimension at least $2\varepsilon(\alpha) + 2$, for some $\varepsilon(\alpha) \in \{0, 1\}$.

2. The restriction of $\cup : H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ to $L_\alpha \wedge L_\alpha$ has rank $\varepsilon(\alpha)$.

3. If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.

If $M$ is a compact Kähler manifold, then $M$ is formal, and so the theorem applies: each $L_\alpha$ has dimension $2g(\alpha) \geq 4$, and the restriction of the cup-product map to $L_\alpha \wedge L_\alpha$ has rank $\varepsilon(\alpha) = 1$. 

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Theorem

Let $X$ be a 1-formal, quasi-Kähler manifold (for instance, a compact Kähler manifold). Then:

1. $\Omega^1_1(X) = \overline{R^1(X, \mathbb{Q})}$ and $\Omega^1_r(X) \subseteq \sigma_r(R^1(X, \mathbb{Q}))^c$, for $r \geq 2$.
2. If $W^1(X)$ contains no positive-dimensional translated subtori, then $\Omega^1_r(X) = \sigma_r(R^1(X, \mathbb{Q}))^c$, for all $r \geq 1$.

In general, though, this last inclusion can be strict.

Theorem

Let $X$ be a 1-formal, smooth, quasi-projective variety. Suppose

1. $W^1(X)$ has a 1-dimensional component not passing through 1;
2. $R^1(X)$ has no codimension-1 components.

Then $\Omega^1_2(X)$ is strictly contained in $\sigma_2(R^1(X, \mathbb{Q}))^c$.

Concrete example: the complement of the “deleted $B_3$” arrangement.
The Dwyer–Fried sets of a compact Kähler manifold need not be open.

**Example**

- Let $C_1$ be a curve of genus 2 with an elliptic involution $\sigma_1$. Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus 1.

- Let $C_2$ be a curve of genus 3 with a free involution $\sigma_2$. Then $\Sigma_2 = C_2/\sigma_2$ is a curve of genus 2.

- We let $\mathbb{Z}_2$ act freely on the product $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$. The quotient space, $M$, is a smooth, minimal, complex projective surface of general type with $p_g(M) = q(M) = 3$, $K_M^2 = 8$.

- The group $\pi = \pi_1(M)$ can be computed by method due to I. Bauer, F. Catanese, F. Grunewald. Identifying $\pi_{ab} = \mathbb{Z}^6$, $\hat{\pi} = (\mathbb{C}^\times)^6$, get

  $$\mathcal{V}^1(\pi) = \{ t \mid t_1 = t_2 = 1 \} \cup \{ t_4 = t_5 = t_6 = 1, \ t_3 = -1 \}.$$  

- It follows that $\Omega^1_2(\pi)$ is not open.
Proposition ([SYZ])

Suppose $\mathcal{V}^i(X)$ is a union of algebraic subgroups. If $X^\nu$ is a free abelian cover with finite Betti numbers up to degree $i$, then any finite regular abelian cover of $X^\nu$ has the same finiteness property.

For general quasi-projective varieties, the conclusion does not hold.

Example

- The Brieskorn 3-manifold $M = \Sigma(3, 3, 6)$ is the singularity link of a weighted homogeneous polynomial; thus, it has the homotopy type of a smooth (non-formal) quasi-projective variety.

- A shown in [Dimca–Papadima-A.S. 2011], the variety $\mathcal{V}^1(M)$ has 3 positive-dimensional irreducible components, all of dimension 2, none of which passes through the identity.

- It follows that $b_1(\Sigma(3, 3, 6)^{\text{fab}}) < \infty$, while $b_1(\Sigma(3, 3, 6)^{\text{ab}}) = \infty$. 
Hyperplane arrangements

- Let $\mathcal{A}$ be an arrangement of $n$ hyperplanes in $\mathbb{C}^d$, defined by a polynomial $f = \prod_{H \in \mathcal{A}} \alpha_H$, with $\alpha_H$ linear forms.

- The complement, $X = X(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$, is a smooth, quasi-projective variety. It is also a formal space.

- The homology groups $H_*(X, \mathbb{Z})$ are torsion-free.

- The cohomology ring $A = H^*(X, \mathbb{C})$ is the quotient $A = E/I$ of the exterior algebra on $n$ generators, modulo an ideal determined by the intersection lattice $L(\mathcal{A})$.

- The fundamental group $G = \pi_1(X(\mathcal{A}))$ has a presentation associated to a generic plane section, with generators corresponding to the lines, and commutator relators corresponding to the multiple points. In particular, $G_{ab} = \mathbb{Z}^n$. 
Identify \( \hat{G} = H^1(X, \mathbb{C}^\times) = (\mathbb{C}^\times)^n \) and \( H^1(X, \mathbb{C}) = \mathbb{C}^n \).

Set \( \mathcal{V}^i(\mathcal{A}) = \mathcal{V}^i(X) \), etc.

Tangent cone formula holds:

\[
\tau_1(\mathcal{V}^i(\mathcal{A})) = TC_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A}).
\]

Components of \( \mathcal{R}^i(\mathcal{A}) \) are rationally defined linear subspaces of \( \mathbb{C}^n \), depending only on \( L(\mathcal{A}) \).

Components of \( \mathcal{V}^i(\mathcal{A}) \) are subtori of \( (\mathbb{C}^\times)^n \), possibly translated by roots of 1.

Components passing through 1 are combinatorially determined:

\[
L \subset \mathcal{R}^i(\mathcal{A}) \leadsto T = \exp(L) \subset \mathcal{V}^i(\mathcal{A}).
\]

\( \mathcal{V}^1(\mathcal{A}) \) may contain translated subtori.
Example (Braid arrangement $A_3$)

$R^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from triple points), and one non-local component, from neighborly partition $\Pi = (16|25|34)$:

- $L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}$,
- $L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\}$,
- $L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}$,
- $L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\}$,
- $L_\Pi = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}$.

There are no translated components.
Theorem

Suppose \( \mathcal{Y}^k(\mathcal{A}) \) contains no translated components. Then:

1. \( X(\mathcal{A}) \) is \( k \)-straight.
2. \( \Omega^k_r(\mathcal{A}) = \text{Gr}_r(\mathbb{Q}^n) \setminus \sigma_r(\mathcal{R}^k(\mathcal{A}, \mathbb{Q})) \), for all \( 1 \leq r \leq n \).

Proposition

Let \( \mathcal{A} \) be an arrangement of \( n \) lines in \( \mathbb{C}^2 \), and let \( m \) be the maximum multiplicity of its intersection points.

1. If \( m = 2 \), then \( \Omega^1_r(\mathcal{A}) = \text{Gr}_r(\mathbb{Q}^n) \), for all \( r \geq 1 \).
2. If \( m \geq 3 \), then \( \Omega^1_r(\mathcal{A}) = \emptyset \), for all \( r \geq n - m + 2 \).

Proposition

Suppose \( \mathcal{A} \) has 1 or 2 lines which contain all the intersection points of multiplicity 3 and higher. Then \( X(\mathcal{A}) \) is 1-straight, and

\[ \Omega^1_r(\mathcal{A}) = \sigma_r(\mathcal{R}^1(\mathcal{A}, \mathbb{Q}))^c. \]
Let $\mathcal{A}$ be defined by $f = z_0z_1(z_0^2 - z_1^2)(z_0^2 - z_2^2)(z_1^2 - z_2^2)$. Then:

- $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^8$ contains 7 local components (from 6 triple points and 1 quadruple point), and 5 non-local components (from braid sub-arrangements). In particular, $\text{codim} \mathcal{R}^1(\mathcal{A}) = 5$.

- In addition to the corresponding 12 subtori, $\mathcal{V}^1(\mathcal{A}) \subset (\mathbb{C}^\times)^8$ also contains $\rho \cdot T$, where $T \cong \mathbb{C}^\times$, and $\rho$ is a root of unity of order 2.

- Thus, the complement $\mathcal{X}$ is not 1-straight.

- But $\mathcal{X}$ is formal, so $\Omega^1_2(\mathcal{A})$ is strictly contained in $\sigma_2(\mathcal{R}^1(\mathcal{A}))^c$. 
Milnor fibration

- Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^d$, defined by a polynomial $f = \alpha_1 \cdots \alpha_n$.
- Milnor fibration: $f: \mathbb{C}^d \setminus V(f) \to \mathbb{C} \setminus \{0\}$.
- Milnor fiber: $F = f^{-1}(1)$, a smooth, affine variety, with the homotopy type of a $(d - 1)$-dimensional, finite CW-complex (not necessarily formal: H. Zuber 2010).
- $F$ is a Galois, $\mathbb{Z}$-cover of $X = \mathbb{C}^d \setminus V(f)$; it is also a Galois, $\mathbb{Z}_n$-cover of $U = \mathbb{CP}^{d-1} \setminus V(f)$.
- Hence, we may compute $H_1(F, \mathbb{k})$ by counting certain torsion points on the varieties $V_j^1(U, \mathbb{k})$, provided $\text{char} \mathbb{k} \nmid n$.
- Let $s = (s_1, \ldots, s_n)$ be positive integers with $\gcd(s) = 1$. The polynomial $f_s = \alpha_1^{s_1} \cdots \alpha_n^{s_n}$ defines a multi-arrangement $\mathcal{A}_s$, with $X(\mathcal{A}_s) = X(\mathcal{A})$, but $F(\mathcal{A}_s) \not\cong F(\mathcal{A})$, in general.
**Question (Dimca–Némethi 2002)**

Let \( f : \mathbb{C}^d \to \mathbb{C} \) be a homogeneous polynomial, \( X = \mathbb{C}^d \setminus V(f) \), and \( F = f^{-1}(1) \). If \( H_*(X, \mathbb{Z}) \) is torsion-free, is \( H_*(F, \mathbb{Z}) \) also torsion-free?


Not for \( H_1(F(A_s), \mathbb{Z}) \), nor for \( H_*(F(A), \mathbb{Z}) \).

**Example**

Take \( A \) to be the deleted \( B_3 \) arrangement, with weights \( s = (2, 1, 3, 3, 2, 2, 1, 1) \), so that

\[
f_s = z_0^2 z_1 (z_0^2 - z_1^2)^3 (z_0^2 - z_2^2)^2 (z_1^2 - z_2^2).
\]

Then \( \dim_k H_1(F(A_s), k) = 7 \) if \( \text{char } k \neq 2, 3, 5 \), yet \( \dim_k H_1(F(A_s), k) = 9 \) if \( \text{char } k = 2 \). In fact:

\[
H_1(F(A_s), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2
\]
Example

Let $\mathcal{A}$ be the arrangement of 24 hyperplanes in $\mathbb{C}^8$, defined by

$$f = z_1 z_2 (z_1^2 - z_2^2) (z_1^2 - z_3^2) (z_2^2 - z_3^2) y_1 y_2 y_3 y_4 y_5 (z_1 - y_1) (z_1 - y_2) \cdot (z_1^2 - 4 y_1^2) (z_1 - y_3) (z_1^2 - y_4^2) (z_1 - 2 y_4) (z_1^2 - y_5^2) (z_1 - 2 y_5).$$

The 2-torsion part of $H_6(F(\mathcal{A}), \mathbb{Z})$ is $(\mathbb{Z}_2)^{54}$.

Question

Are any of the following determined by the intersection lattice $L(\mathcal{A})$:

1. The translated components in $\mathcal{V}^k(\mathcal{A})$.
2. The Dwyer–Fried sets $\Omega^i_r(\mathcal{A})$.
3. The Betti numbers of $F(\mathcal{A})$.
4. The torsion in $H_*(F(\mathcal{A}), \mathbb{Z})$. 


