Hyperplane arrangements, Milnor fibrations, and boundary manifolds

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An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.

*Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.

*Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.

The Boolean arrangement $\mathcal{B}_n$

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^\ast)^n$.

The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type $A_{n-1}$)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1, \ldots, n\}$.
- $M(\mathcal{A}_n)$: configuration space of $n$ ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on $n$ strings).
**Figure**: A planar slice of the braid arrangement $\mathcal{A}_4$
We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

Fix an ordering $\mathcal{A} = \{H_1, \ldots, H_n\}$, and choose linear forms $f_i : \mathbb{C}^{\ell} \to \mathbb{C}$ with $\ker(f_i) = H_i$.

Define an injective linear map
\[
\iota : \mathbb{C}^{\ell} \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \ldots, f_n(z)).
\]

This map restricts to an inclusion $\iota : M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence,
\[
M(\mathcal{A}) = \iota(\mathbb{C}^{\ell}) \cap (\mathbb{C}^*)^n,
\]
a “very affine” subvariety of $(\mathbb{C}^*)^n$, and thus, a Stein manifold.

Therefore, $M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension $\ell$. 
In fact, $M = M(\mathcal{A})$ admits a minimal cell structure (Dimca and Papadima 2003). Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.

The Betti numbers $b_q(M) := \text{rank } H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M)t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\text{rank}(X)},$$

where $\mu: L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathcal{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y \supseteq X} \mu(Y)$.

The Orlik–Solomon algebra $H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra on generators $\{e_H \mid H \in \mathcal{A}\}$ by an ideal determined by the circuits in the matroid of $\mathcal{A}$.

Thus, the ring $H^*(M, \mathbb{k})$ is determined by $L(\mathcal{A})$, for every field $\mathbb{k}$. 
Let $X$ be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

Let $k$ be an algebraically closed field, and let $\text{Hom}(\pi, k^*)$ be the affine algebraic group of $k$-valued, multiplicative characters on $\pi$.

The characteristic varieties of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$V^q_s(X, k) = \{ \rho \in \text{Hom}(\pi, k^*) \mid \dim_k H^q(X, k\rho) \geq s \}.$$ 

Here, $k\rho$ is the local system defined by $\rho$, i.e, $k$ viewed as a $k\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, k\rho) = H_i(C_*(\tilde{X}, k) \otimes_{k\pi} k\rho)$.

These loci are Zariski closed subsets of the character group.
Let $A = H^*(X, k)$. If $\text{char } k = 2$, assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$.

Thus, we get a cochain complex

$$(A, \cdot a) : A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{\ldots},$$

known as the Aomoto complex of $A$.

The resonance varieties of $X$ are the jump loci for the Aomoto-Betti numbers

$$\mathcal{R}_s^q(X, k) = \{ a \in A^1 \mid \dim_k H^q(A, \cdot a) \geq s \},$$

These loci are homogeneous subvarieties of $A^1 = H^1(X, k)$. 
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^3$, and identify $H^1(M(\mathcal{A}), k) = k^n$, with basis dual to the meridians.

The resonance varieties $R^1_s(\mathcal{A}, k) := R^1_s(M(\mathcal{A}), k) \subset k^n$ lie in the hyperplane $\{x \in k^n \mid x_1 + \cdots + x_n = 0\}$.

$R^1(\mathcal{A}) = R^1_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $\mathbb{C}^n$.

Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

$R^1_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$. 
Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geq 3$ gives rise to a local component of $R^1(\mathcal{A})$, of dimension $k - 1$.

More generally, every $k$-multinet on a sub-arrangement $B \subseteq \mathcal{A}$ gives rise to a component of dimension $k - 1$, and all components of $R^1(\mathcal{A})$ arise in this way.

The resonance varieties $R^1(\mathcal{A}, k)$ can be more complicated, e.g., they may have non-linear components.
Example (Braid arrangement $A_4$)

$R^1(A) \subset \mathbb{C}^6$ has 4 components coming from the triple points, and one component from the above 3-net:

\[
\begin{align*}
L_{124} &= \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}, \\
L_{135} &= \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\}, \\
L_{236} &= \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}, \\
L_{456} &= \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\}, \\
L &= \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.
\end{align*}
\]
• Let $\text{Hom}(\pi_1(M), k^*) = (k^*)^n$ be the character torus.

• The characteristic variety $\mathcal{V}^1(\mathcal{A}, k) := \mathcal{V}^1_1(M(\mathcal{A}), k) \subset (k^*)^n$ lies in the subtorus $\{t \in (k^*)^n \mid t_1 \cdots t_n = 1\}$.

• $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}^1(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.

• If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}^1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}^1(\mathcal{A})$.

• All components of $\mathcal{V}^1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

• In general, though, there are translated subtori in $\mathcal{V}^1(\mathcal{A})$. 


**Theorem (Denham, S., Yuzvinsky 2014)**

Let $\mathcal{A}$ be a central, essential hyperplane arrangement in $\mathbb{C}^n$ with complement $M = M(\mathcal{A})$. Suppose $\mathcal{A} = \mathbb{Z}[\pi]$ or $\mathcal{A} = \mathbb{Z}[\pi_{ab}]$. Then $H^p(M, A) = 0$ for all $p \neq n$, and $H^n(M, A)$ is a free abelian group.

**Corollary**

1. $M$ is a duality space of dimension $n$ (due to Davis, Januszkiewicz, Okun 2011).
2. $M$ is an abelian duality space of dimension $n$.
3. The characteristic and resonance varieties of $M$ propagate:

\[
\mathcal{V}_1^1(M, k) \subseteq \cdots \subseteq \mathcal{V}_n^1(M, k)
\]

\[
\mathcal{R}_1^1(M, k) \subseteq \cdots \subseteq \mathcal{R}_n^1(M, k)
\]
For each $H \in \mathcal{A}$, let $f_H : \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel $H$.

For each choice of multiplicities $m = (m_H)_{H \in \mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.

The map $Q_m : \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m : \mathcal{M}(\mathcal{A}) \to \mathbb{C}^*.$

This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m),$

$$F_m(\mathcal{A}) \xrightarrow{F_m} \mathcal{M}(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$
The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.

$F_m(\mathcal{A})$ has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, and of dimension $\ell - 1$.

The *(geometric) monodromy* is the diffeomorphism

$$h: F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i/N}z.$$

If all $m_H = 1$, the polynomial $Q = Q_m(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

**Example**

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then:

- $M(\mathcal{A}) = \mathbb{C}^*$.
- $Q_m(\mathcal{A}) = z^m$.
- $F_m(\mathcal{A}) = m$-roots of 1.
**Example**

Let \( \mathcal{A} \) be a pencil of 3 lines through the origin of \( \mathbb{C}^2 \). Then \( F(\mathcal{A}) \) is a thrice-punctured torus, and \( h \) is an automorphism of order 3:

More generally, if \( \mathcal{A} \) is a pencil of \( n \) lines in \( \mathbb{C}^2 \), then \( F(\mathcal{A}) \) is a Riemann surface of genus \((n-1)/2\), with \( n \) punctures.
Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential arrangement. The inclusion $\iota: M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$
Homology of the Milnor Fiber

- Assume $\gcd(m) = 1$. Then $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U(\mathcal{A}) = \mathbb{P}(\mathcal{M}(\mathcal{A}))$ defined by the homomorphism

$$\delta_m: \pi_1(U(\mathcal{A})) \to \mathbb{Z}_N, \quad x_H \mapsto m_H \mod N$$

- Let $\widehat{\delta}_m: \text{Hom}(\mathbb{Z}_N, \mathbb{k}^*) \to \text{Hom}(\pi_1(U(\mathcal{A})), \mathbb{k}^*)$. If $\text{char}(\mathbb{k}) \nmid N$, then

$$\dim_{\mathbb{k}} H_q(F_m(\mathcal{A}), \mathbb{k}) = \sum_{s \geq 1} \left| \mathcal{V}_s^q(U(\mathcal{A}), \mathbb{k}) \cap \text{im}(\widehat{\delta}_m) \right|.$$ 

- This gives a formula for the characteristic polynomial

$$\Delta^\mathbb{k}_q(t) = \det(t \cdot \text{id} - h_*)$$

of the algebraic monodromy, $h_*: H_q(F(\mathcal{A}), \mathbb{k}) \to H_q(F(\mathcal{A}), \mathbb{k})$, in terms of the characteristic varieties of $U(\mathcal{A})$ and multiplicities $m$. 
Let $\Delta = \Delta_1^C$, and write

$$\Delta(t) = \prod_{d|n} \Phi_d(t)^{e_d(A)},$$

where $\Phi_d(t)$ is the $d$-th cyclotomic polynomial, and $e_d(A) \in \mathbb{Z}_{\geq 0}$.

Not all divisors of $n$ appear in $(\ast)$. For instance, if $d \nmid |A_X|$, for some $X \in L_2(A)$, then $e_d(A) = 0$ (Libgober 2002).

In particular, if $L_2(A)$ has only flats of multiplicity 2 and 3, then $\Delta(t) = (t - 1)^{n-1}(t^2 + t + 1)^{e_3}$.

If multiplicity 4 appears, then also get factor of $(t + 1)^{e_2} \cdot (t^2 + 1)^{e_4}$.

Question: Is $\Delta(t)$ determined by $L(A)$?
**Theorem (Papadima–S. 2014)**

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta_A(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

The combinatorial quantities involved in this theorem (and its generalizations) are

$$\beta_p(\mathcal{A}) = \dim_k H^1(A, \sigma),$$

where $A = H^*(M(\mathcal{A}), k)$, with $\text{char}(k) = p$, and $\sigma = \sum_{H \in A} e_H \in A^1$.

**Conjecture (PS)**

Let $\mathcal{A}$ be an arrangement of rank at least 3. Then $e_{p^s}(\mathcal{A}) = 0$, for all primes $p$ and integers $s \geq 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$
TORSION IN HOMOLOGY


For every prime \( p \geq 2 \), there is a multi-arrangement \((\mathcal{A}, m)\) such that \( H_1(F_m(\mathcal{A}), \mathbb{Z}) \) has non-zero \( p \)-torsion.

Simplest example: the arrangement of 8 hyperplanes in \( \mathbb{C}^3 \) with

\[
Q_m(\mathcal{A}) = x^2 y (x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)
\]

Then \( H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).
We now can generalize and reinterpret these examples, as follows.

**Theorem (Denham–S. 2014)**

Suppose $\mathcal{A}$ admits a ‘pointed’ multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\setminus\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}^1(\mathcal{A}', \mathbb{k})$ varies with $\text{char}(\mathbb{k})$. 
To produce \( p \)-torsion in the homology of the usual Milnor fiber, we use a ‘polarization’ construction:

\[
(A, m) \mapsto A \parallel m,\text{ an arrangement of } N = \sum_{H \in A} m_H \text{ hyperplanes, of rank equal to } \text{rank } A + |\{H \in A : m_H \geq 2\}|.
\]

**Theorem (DS)**

Suppose \( A \) admits a pointed multinet, with distinguished hyperplane \( H \) and multiplicity \( m \). Let \( p \) be a prime dividing \( m_H \).

There is then a choice of multiplicities \( m' \) on the deletion \( A' = A \backslash \{H\} \) such that \( H_q(F(B), \mathbb{Z}) \) has \( p \)-torsion, where \( B = A' \parallel m' \) and 
\[
q = 1 + |\{K \in A' : m'_K \geq 3\}|.
\]
**Corollary (DS)**

For every prime $p \geq 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion, for some $q > 1$.

Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1w_2w_3w_4w_5(x^2 - w_1^2)(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1).$$

$$((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)((x - z)^2 - w_4^2)((x - z)^2 - 2w_4^2) \cdot ((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).
The boundary manifold of an arrangement

Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$ ($d \geq 1$).

Let $\mathcal{P}(\mathcal{A}) = \{\mathcal{P}(H)\}_{H \in \mathcal{A}}$, and let $\nu(W)$ be a regular neighborhood of the algebraic hypersurface $W = \bigcup_{H \in \mathcal{A}} \mathcal{P}(H)$ inside $\mathbb{C}\mathbb{P}^d$.

Let $\overline{U} = \mathbb{C}\mathbb{P}^d \setminus \text{int}(\nu(W))$ be the exterior of $\mathcal{P}(\mathcal{A})$.

The boundary manifold of $\mathcal{A}$ is $\partial \overline{U} = \partial \nu(W)$: a compact, orientable, smooth manifold of dimension $2d - 1$.

**Example**

Let $\mathcal{A}$ be a pencil of $n$ hyperplanes in $\mathbb{C}^{d+1}$, defined by $Q = z_1^n - z_2^n$. If $n = 1$, then $\partial \overline{U} = S^{2d-1}$. If $n > 1$, then $\partial \overline{U} = \#^{n-1}S^1 \times S^{2(d-1)}$.

**Example**

Let $\mathcal{A}$ be a near-pencil of $n$ planes in $\mathbb{C}^3$, defined by $Q = z_1(z_2^{n-1} - z_3^{n-1})$. Then $\partial \overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \#^g S^1 \times S^1$. 
By Lefschetz duality: $H_q(\partial \overline{U}, \mathbb{Z}) \cong H_q(U, \mathbb{Z}) \oplus H_{2d-q-1}(U, \mathbb{Z})$

Let $A = H^*(U, \mathbb{Z})$; then $\hat{A} = \text{Hom}_\mathbb{Z}(A, \mathbb{Z})$ is an $A$-bimodule, with $(a \cdot f)(b) = f(ba)$ and $(f \cdot a)(b) = f(ab)$.

**Theorem (Cohen–S. 2006)**

The ring $\hat{A} = H^*(\partial \overline{U}, \mathbb{Z})$ is the “double” of $A$, that is: $\hat{A} = A \oplus \hat{A}$, with multiplication given by $(a, f) \cdot (b, g) = (ab, ag + fb)$, and grading $\hat{A}^q = A^q \oplus \hat{A}^{2d-q-1}$.

Now assume $d = 2$. Then $\partial \overline{U}$ is a graph-manifold of dimension 3, modeled on a graph $\Gamma$ based on the poset $L_{\leq 2}(\mathcal{A})$.

**Theorem (Cohen–S. 2008)**

The manifold $\partial \overline{U}$ admits a minimal cell structure. Moreover,

$$\mathcal{V}_1(\partial \overline{U}) = \bigcup_{v \in V(\Gamma): d_v \geq 3} \{t_v - 1 = 0\},$$

where $d_v$ denotes the degree of the vertex $v$, and $t_v = \prod_{i \in v} t_i$. 
THE BOUNDARY OF THE MILNOR FIBER

Let \((\mathcal{A}, m)\) be a multi-arrangement in \(C^{d+1}\).

Define \(\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap D^{2(d+1)}\) to be the closed Milnor fiber of \((\mathcal{A}, m)\). Clearly, \(F_m(\mathcal{A})\) deform-retracts onto \(\overline{F}_m(\mathcal{A})\).

The boundary of the Milnor fiber of \((\mathcal{A}, m)\) is the compact, smooth, orientable, \((2d - 1)\)-manifold \(\partial \overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap S^{2d+1}\).

The pair \((\overline{F}_m, \partial \overline{F}_m)\) is \((d - 1)\)-connected. In particular, if \(d \geq 2\), then \(\partial \overline{F}_m\) is connected, and \(\pi_1(\partial \overline{F}_m) \to \pi_1(\overline{F}_m)\) is surjective.

**Figure**: Closed Milnor fiber for \(Q(\mathcal{A}) = xy\)
**Example**

- Let $B_n$ be the Boolean arrangement in $\mathbb{C}^n$. Recall $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$, and so $\partial \overline{F} = T^{n-1} \times S^{n-2}$.

- Let $\mathcal{A}$ be a near-pencil of $n$ planes in $\mathbb{C}^3$. Then $\partial \overline{F} = S^1 \times \Sigma_{n-2}$.

The Hopf fibration $\pi : \mathbb{C}^{d+1} \setminus \{0\} \to \mathbb{C}P^d$ restricts to regular, cyclic $n$-fold covers, $\pi : \overline{F} \to \overline{\mathcal{U}}$ and $\pi : \partial \overline{F} \to \partial \overline{\mathcal{U}}$, which fit into the ladder diagram:

```
\[\begin{array}{cccccc}
\mathbb{Z}_n & \rightarrow & \mathbb{Z}_n & \rightarrow & \mathbb{Z}_n & \rightarrow & \mathbb{C}^* & \rightarrow & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial \overline{F} & \rightarrow & \overline{F} & \sim & F & \rightarrow & M & \rightarrow & \mathbb{C}^{d+1} \setminus \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial \overline{\mathcal{U}} & \rightarrow & \overline{\mathcal{U}} & \sim & \mathcal{U} & \equiv & \mathcal{U} & \rightarrow & \mathbb{C}P^d \\
\end{array}\]```
Assume now that \( d = 2 \). The group \( \pi_1(\partial \overline{U}) \) has generators \( x_1, \ldots, x_{n-1} \) corresponding to the meridians around the first \( n - 1 \) lines in \( \mathbb{P}(A) \), and generators \( y_1, \ldots, y_s \) corresponding to the cycles in the associated graph \( \Gamma \).

**Proposition** (S. 2014)

The \( \mathbb{Z}_n \)-cover \( \pi : \partial F \to \partial \overline{U} \) is classified by the homomorphism \( \pi_1(\partial \overline{U}) \to \mathbb{Z}_n \) given by \( x_i \mapsto 1 \) and \( y_j \mapsto 0 \).

**Example**

Let \( A \) be a pencil of \( n + 1 \) planes in \( \mathbb{C}^3 \). Since \( \partial \overline{U} = \#^n S^1 \times S^2 \), and \( \partial F \to \partial \overline{U} \) is a cover with \( n + 1 \) sheets, we see that \( \partial F = \#^{n^2} S^1 \times S^2 \).
**Theorem (Némethi–Szilárd 2012)**

Let $\mathcal{A}$ be an arrangement of $n$ planes in $\mathbb{C}^3$. The characteristic polynomial of the algebraic monodromy acting on $H_1(\partial \overline{F}, \mathbb{C})$ is given by

$$
\Delta(t) = \prod_{X \in L_2(\mathcal{A})} (t - 1)(t^{\gcd(\mu(X) + 1, n)} - 1)^{\mu(X) - 1}.
$$

- This shows that $b_1(\partial \overline{F})$ is a much less subtle invariant than $b_1(F)$: it depends only on the number and type of multiple points of $\mathbb{P}(\mathcal{A})$, but not on their relative position.

- On the other hand, the torsion in $H_1(\partial \overline{F}, \mathbb{Z})$ is still not understood.

- For a generic arrangement of $n$ planes in $\mathbb{C}^3$, I expect that $H_1(\partial \overline{F}, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2} \oplus \mathbb{Z}_n^{(n-2)(n-3)/2}$.

- In general, it would be interesting to see whether all the torsion in $H_1(\partial \overline{F}(\mathcal{A}), \mathbb{Z})$ consists of $\mathbb{Z}_n$-summands, where $n = |\mathcal{A}|$. 
REFERENCES


