Topology and combinatorics of Milnor fibrations of hyperplane arrangements

Alex Suciu
Northeastern University

Conference on Hyperplane Arrangements and Characteristic Classes
Research Institute for Mathematical Sciences, Kyoto
November 13, 2013


HYPERPLANE ARRANGEMENTS

- **A**: A (central) arrangement of hyperplanes in $\mathbb{C}^\ell$.

- Intersection lattice: $L(A)$.

- Complement: $M(A) = \mathbb{C}^\ell \setminus \bigcup_{H \in A} H$.

- The Boolean arrangement $B_n$:
  - $B_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(B_n)$: lattice of subsets of $\{0, 1\}^n$.
  - $M(B_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

- The braid arrangement $A_n$ (or, reflection arr. of type $A_{n-1}$):
  - $A_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(A_n)$: lattice of partitions of $[n] = \{1, \ldots, n\}$.
  - $M(A_n)$: configuration space of $n$ ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on $n$ strings).
Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.

- $L_1(\mathcal{A}) \longleftrightarrow$ lines of $\bar{\mathcal{A}}$, $L_2(\mathcal{A}) \longleftrightarrow$ intersection points of $\bar{\mathcal{A}}$.
- Poset structure of $L_{\leq 2}(\mathcal{A}) \longleftrightarrow$ incidence structure of $\bar{\mathcal{A}}$.

- A flat $X \in L_2(\mathcal{A})$ has multiplicity $q$ if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \supset H\}$ has size $q$, i.e., there are exactly $q$ lines from $\bar{\mathcal{A}}$ passing through $\bar{X}$.
If $\mathcal{A}$ is essential, then $M = M(\mathcal{A})$ is a (very affine) subvariety of $(\mathbb{C}^\times)^n$, where $n = |\mathcal{A}|$.

$M$ has the homotopy type of a connected, finite CW-complex of dimension $\ell$. In fact, $M$ admits a minimal cell structure.

In particular, $H_\bullet(M, \mathbb{Z})$ is torsion-free. The Betti numbers $b_q(M) := \text{rank } H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\text{rank}(X)}.$$ 

The Orlik–Solomon algebra $A = H^\bullet(M, \mathbb{Z})$ is the quotient of the exterior algebra on generators dual to the meridians, by an ideal determined by the circuits in the matroid of $\mathcal{A}$.

On the other hand, the group $\pi_1(M)$ is not determined by $L(\mathcal{A})$. 
For each $H \in \mathcal{A}$, let $f_H: \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel $H$.

Let $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$, a homogeneous polynomial of degree $n$.

The map $Q: \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q: M(\mathcal{A}) \to \mathbb{C}^\ast$.

This is the projection of a smooth, locally trivial bundle, known as the Milnor fibration of the arrangement.

The typical fiber, $F(\mathcal{A}) = Q^{-1}(1)$, is a very affine variety, with the homotopy type of a connected, finite CW-complex of dim $\ell - 1$.

The monodromy of the bundle is the diffeomorphism

$$h: F \to F, \quad z \mapsto e^{2\pi i/n} z.$$
**Example**

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and $h$ is an automorphism of order 3:

More generally, if $\mathcal{A}$ is a pencil of $n$ lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\left(\frac{n-1}{2}\right)$, with $n$ punctures.

**Example**

Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q = z_1 \cdots z_n$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and $F(\mathcal{B}_n) = \ker(Q) \cong (\mathbb{C}^*)^{n-1}$.
Two basic questions about the Milnor fibration of an arrangement:

(Q1) Are the Betti numbers $b_q(F(\mathcal{A}))$ and the characteristic polynomial of the algebraic monodromy, $h_q : H_q(F(\mathcal{A}), \mathbb{C}) \to H_q(F(\mathcal{A}), \mathbb{C})$, determined by $L(\mathcal{A})$?

(Q2) Are the homology groups $H_\ast(F(\mathcal{A}), \mathbb{Z})$ torsion-free? If so, does $F(\mathcal{A})$ admit a minimal cell structure?

Recent progress on both questions:
- A partial, positive answer to (Q1).
- A negative answer to (Q2).
Let $\Delta_A(t) := \det(h_1 - t \cdot \text{id})$. Then $b_1(F(A)) = \deg \Delta_A$.

**Theorem (Papadima–S. 2013)**

Suppose all flats $X \in L_2(A)$ have multiplicity 2 or 3. Then $\Delta_A(t)$, and thus $b_1(F(A))$, are combinatorially determined.

**Theorem (Denham–S. 2013)**

For every prime $p \geq 2$, there is an arrangement $A$ such that $H_q(F(A), \mathbb{Z})$ has non-zero $p$-torsion, for some $q > 1$.

- In both results, we relate the cohomology jump loci of $M(A)$ in characteristic $p$ with those in characteristic 0.
- In the first result, the bridge between the two goes through the representation variety $\text{Hom}_{\text{Lie}}(h(A), sl_2)$.
- A key combinatorial ingredient in both proofs is the notion of multinet.
Let \( A = H^*(M(A), \mathbb{k}) \) — an algebra that depends only on \( L(A) \) (and the field \( \mathbb{k} \)).

For each \( a \in A^1 \), we have \( a^2 = 0 \). Thus, we get a cochain complex, \((A, \cdot a)\): \( A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \rightarrow \cdots \)

The (degree 1) resonance varieties of \( A \) are the cohomology jump loci of this “Aomoto complex”:

\[
\mathcal{R}_s(A, \mathbb{k}) = \{ a \in A^1 \mid \dim_{\mathbb{k}} H^1(A, \cdot a) \geq s \},
\]

In particular, \( a \in A^1 \) belongs to \( \mathcal{R}_1(A, \mathbb{k}) \) iff there is \( b \in A^1 \) not proportional to \( a \), such that \( a \cup b = 0 \) in \( A^2 \).
Now assume \( k \) has characteristic \( p > 0 \).

Let \( \sigma = \sum_{H \in \mathcal{A}} e_H \in A^1 \) be the “diagonal” vector, and define

\[
\beta_p(A) = \dim_k H^1(A, \cdot \sigma).
\]

That is, \( \beta_p(A) = \max\{s \mid \sigma \in R^1_s(A, k)\} \).

Clearly, \( \beta_p(A) \) depends only on \( L(A) \) and \( p \). Moreover, \( 0 \leq \beta_p(A) \leq |A| - 2 \).

**Theorem (PS)**

*If \( L_2(A) \) has no flats of multiplicity \( 3r \) with \( r > 1 \), then \( \beta_3(A) \leq 2 \).*

For each \( m \geq 1 \), there is a matroid \( \mathcal{M}_m \) with all rank 2 flats of multiplicity 3, and such that \( \beta_3(\mathcal{M}_m) = m \).

\( \mathcal{M}_1 \): pencil of 3 lines. \( \mathcal{M}_2 \): Ceva arrangement.

\( \mathcal{M}_m \) with \( m > 2 \): not realizable over \( \mathbb{C} \).
The monodromy $h: F(\mathcal{A}) \to F(\mathcal{A})$ has order $n = |\mathcal{A}|$. Thus,

$$\Delta_\mathcal{A}(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$

where $\Phi_1 = t - 1$, $\Phi_2 = t + 1$, $\Phi_3 = t^2 + t + 1$, $\Phi_4 = t^2 + 1$, ... are the cyclotomic polynomials, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geq 0}$.

- Easy to see: $e_1(\mathcal{A}) = n - 1$. Hence, $H_1(F(\mathcal{A}), \mathbb{C})$, when viewed as a module over $\mathbb{C}[\mathbb{Z}_n]$, decomposes as

$$(\mathbb{C}[t]/(t-1))^{n-1} \oplus \bigoplus_{1 < d | n} (\mathbb{C}[t]/\Phi_d(t))^{e_d(\mathcal{A})}.$$

- In particular, $b_1(F(\mathcal{A})) = n - 1 + \sum_{1 < d | n} \varphi(d) e_d(\mathcal{A})$. 
Thus, in degree 1, question (Q1) is equivalent to: are the integers $e_d(A)$ determined by $L_{\leq 2}(A)$?

Not all divisors of $n$ appear in the above formulas: If $d$ does not divide $|A_X|$, for some $X \in L_2(A)$, then $e_d(A) = 0$ (Libgober).

In particular, if $L_2(A)$ has only flats of multiplicity 2 and 3, then $\Delta_A(t) = (t - 1)^{n-1}(t^2 + t + 1)^{e_3}$.

If multiplicity 4 appears, then also get factor of $(t + 1)^{e_2} \cdot (t^2 + 1)^{e_4}$.


$e_{ps}(A) \leq \beta_p(A)$, for all $s \geq 1$. 
**Theorem (PS13)**

*Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.*

A similar result holds for $e_2(\mathcal{A})$ and $e_4(\mathcal{A})$, under some additional hypothesis.

**Corollary**

*If $\bar{\mathcal{A}}$ is an arrangement of $n$ lines in $\mathbb{P}^2$ with only double and triple points, then $\Delta_{\mathcal{A}}(t) = (t - 1)^{n-1}(t^2 + t + 1)\beta_3(\bar{\mathcal{A}})$ is combinatorially determined.*

**Corollary (Libgober 2012)**

*If $\bar{\mathcal{A}}$ is an arrangement of $n$ lines in $\mathbb{P}^2$ with only double and triple points, then the question whether $\Delta_{\mathcal{A}}(t) = (t - 1)^{n-1}$ or not is combinatorially determined.*
**Conjecture**

Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^3$. Then

$$\Delta_{\mathcal{A}}(t) = (t - 1)^{|\mathcal{A}| - 1}(t^2 + t + 1)^{\beta_3(\mathcal{A})}[(t + 1)(t^2 + 1)]^{\beta_2(\mathcal{A})}.$$
**MULTINETS**

**Definition (Falk and Yuzvinsky)**

A *multinet* on $A$ is a partition of the set $A$ into $k \geq 3$ subsets $A_1, \ldots, A_k$, together with an assignment of multiplicities, $m: A \to \mathbb{N}$, and a subset $X \subseteq L_2(A)$, called the base locus, such that:

1. There is an integer $d$ such that $\sum_{H \in A_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If $H$ and $H'$ are in different classes, then $H \cap H' \in X$.
3. For each $X \in X$, the sum $n_X = \sum_{H \in A_\alpha : H \supseteq X} m_H$ is independent of $\alpha$.
4. Each set $(\bigcup_{H \in A_\alpha} H) \setminus X$ is connected.

- A similar definition can be made for any (rank 3) matroid.
- A multinet as above is also called a $(k, d)$-multinet, or a $k$-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in A$. 
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.
- In this case, $|\mathcal{A}_\alpha| = |\mathcal{A}| / k = d$, for all $\alpha$.
- Moreover, $\mathcal{X}$ has size $d^2$, and is encoded by a $(k - 2)$-tuple of orthogonal Latin squares.

A $(3, 2)$-net on the $A_3$ arrangement $\mathcal{X}$ consists of 4 triple points ($n_X = 1$)

A $(3, 4)$-multinet on the $B_3$ arrangement $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 triple points ($n_X = 2$)
A $(3, 3)$-net on the Ceva matroid. A $(4, 3)$-net on the Hessian matroid.
• If $A$ has no flats of multiplicity $kr$, for some $r > 1$, then every reduced $k$-multinet is a $k$-net.

• (Kawahara): given any Latin square, there is a matroid $M$ with a 3-net $(M_1, M_2, M_3)$ realizing it, such that each $M_\alpha$ is uniform.

• (Yuzvinsky and Pereira–Yuz): If $A$ supports a $k$-multinet with $|X| > 1$, then $k = 3$ or 4; if the multinet is not reduced, then $k = 3$.

• (Wakefield & al): The only $(4, 3)$-net in $\mathbb{CP}^2$ is the Hessian; there are no $(4, 4), (4, 5)$, or $(4, 6)$ nets in $\mathbb{CP}^2$.

• Conjecture (Yuz): The only 4-multinet is the Hessian $(4, 3)$-net.
Lemma (PS)

If $A$ supports a $3$-net with parts $A_\alpha$, then:

1. $1 \leq \beta_3(A) \leq \beta_3(A_\alpha) + 1$, for all $\alpha$.
2. If $\beta_3(A_\alpha) = 0$, for some $\alpha$, then $\beta_3(A) = 1$.
3. If $\beta_3(A_\alpha) = 1$, for some $\alpha$, then $\beta_3(A) = 1$ or $2$.

All possibilities do occur:

- Braid arrangement: has a $(3, 2)$-net from the Latin square of $\mathbb{Z}_2$.
  $\beta_3(A_\alpha) = 0$ (\forall \alpha) and $\beta_3(A) = 1$.

- Pappus arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$.
  $\beta_3(A_1) = \beta_3(A_2) = 0$, $\beta_3(A_3) = 1$ and $\beta_3(A) = 1$.

- Ceva arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$.
  $\beta_3(A_\alpha) = 1$ (\forall \alpha) and $\beta_3(A) = 2$. 
Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Work of Arapura, Falk, Cohen–S., Libgober–Yuz, Falk–Yuz completely describes the varieties $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) = \mathbb{C}^{\left|\mathcal{A}\right|}$.

- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

- $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$. 
Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geq 3$ gives rise to a local component of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$, of dimension $k - 1$.

More generally, every $k$-multinet on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k - 1$, and all components of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ arise in this way.

Note: the varieties $\mathcal{R}_1(\mathcal{A}, \mathbb{k})$ with $\text{char}(\mathbb{k}) > 0$ can be more complicated: components may be non-linear, and they may intersect non-transversely.

**Theorem (PS)**

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ has at least $(3^{\beta_3(\mathcal{A})} - 1)/2$ essential components, all corresponding to $3$-nets.
Let $X$ be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

Let $k$ be an algebraically closed field, and let $\text{Hom}(\pi, k^*) = H^1(X, k^*)$ be the character group of $\pi$.

The (degree 1) characteristic varieties of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$\mathcal{V}_s(X, k) = \{\rho \in \text{Hom}(\pi, k^*) \mid \dim_k H_1(X, k_\rho) \geq s\}.$$

Let $X = M(\mathcal{A})$, and identify $\text{Hom}(\pi, k^*) = (k^*)^n$, where $n = |\mathcal{A}|$.

The characteristic varieties $\mathcal{V}_s(\mathcal{A}, k) := \mathcal{V}_s(M(\mathcal{A}), k)$ lie in the subtorus $\{t \in (k^*)^n \mid t_1 \cdots t_n = 1\}$. 
Work of Arapura, Libgober, Cohen–S., S., Libgober–Yuz, Falk–Yuz, Dimca, Dimca–Papadima–S., Artal–Cogolludo–Matei, Budur–Wang ... provides a fairly explicit description of the varieties $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$:

- Each variety $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.

- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$.

- Moreover, $T = f^*(H^1(S, \mathbb{C}^*))$, where $f: M(\mathcal{A}) \to S$ is an orbifold fibration, with base $S = \mathbb{C}P^1 \setminus \{k \text{ points}\}$, for some $k \geq 3$.

- All components of $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
The Milnor fiber $F(\mathcal{A})$ is a regular $\mathbb{Z}_n$-cover of the projectivized complement $U = M(\mathcal{A})/\mathbb{C}^*$.

This cover classified by the homomorphism $\delta: \pi_1(U) \to \mathbb{Z}_n$ that sends each meridian to 1.

Let $\hat{\delta}: \text{Hom}(\mathbb{Z}_n, k^*) \to \text{Hom}(\pi_1(U), k^*)$. If $\text{char}(k) \nmid n$, then

$$\dim_k H_1(F(\mathcal{A}), k) = \sum_{s \geq 1} \left| \mathcal{V}_s(U, k) \cap \text{im}(\hat{\delta}) \right|.$$ 

The available information on $\mathcal{V}_s(U, \mathbb{C}) \cong \mathcal{V}_s(\mathcal{A}, \mathbb{C})$ implies:

**Theorem**

*If $\mathcal{A}$ admits a reduced $k$-multinet, then $e_k(\mathcal{A}) \geq k - 2$.***
Theorem (PS)

Suppose $L_2(A)$ has no flats of multiplicity $3r$ with $r > 1$. Then, the following conditions are equivalent:

1. $L_{\leq 2}(A)$ admits a reduced 3-multinet.
2. $L_{\leq 2}(A)$ admits a 3-net.
3. $\beta_3(A) \neq 0$.
4. $e_3(A) \neq 0$.

Moreover, $\beta_3(A) \leq 2$ and $\beta_3(A) = e_3(A)$.

- (2) $\Rightarrow$ (1): obvious.
- (1) $\Rightarrow$ (4): by above theorem.
- (4) $\Rightarrow$ (3): by modular bound $e_p(A) \leq \beta_p(A)$.
- (3) $\Rightarrow$ (2): use flat, $sl_2$-valued connections on the OS-algebra.
- $\beta_3(A) \leq 2$: a previous theorem.
- Last assertion: put things together, and use [ACM].
Let \((\mathcal{A}, m)\) be a multi-arrangement, with defining polynomial

\[ Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H}, \]

Let \(F_m(\mathcal{A}) = Q_m^{-1}(1)\) be the corresponding Milnor fiber.

**Theorem (Cohen–Denham–S. 2003)**

For every prime \(p \geq 2\), there is a multi-arrangement \((\mathcal{A}, m)\) such that \(H_1(F_m(\mathcal{A}), \mathbb{Z})\) has non-zero \(p\)-torsion.

Simplest example: the arrangement of 8 hyperplanes in \(\mathbb{C}^3\) with

\[ Q_m(\mathcal{A}) = x^2y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2) \]

Then \(H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\).
We now can generalize and reinterpret these examples, as follows.

A **pointed multinet** on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H | n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

**Theorem (Denham–S. 2013)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1(\mathcal{A}', \mathbb{k})$ varies with $\text{char}(\mathbb{k})$. 
To produce $p$-torsion in the homology of the usual Milnor fiber, we use a “polarization” construction:

$$(\mathcal{A}, m) \leadsto \mathcal{A} \parallel m,$$ an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\text{rank } \mathcal{A} + |\{H \in \mathcal{A} : m_H \geq 2\}|$.

**Theorem (DS)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has $p$-torsion, where $\mathcal{B} = \mathcal{A}' \parallel m'$ and $q = 1 + \left|\{K \in \mathcal{A}' : m'_K \geq 3\}\right|$. 
Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with defining polynomial

$$Q(A) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1 w_2 w_3 w_4 w_5 (x^2 - w_1^2)(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1).$$

$$((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)((x - z)^2 - w_4^2)((x - z)^2 - 2w_4^2) \cdot ((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(A), \mathbb{Z})$ has 2-torsion (of rank 108).