HYPERPLANE ARRANGEMENTS AND
MILNOR FIBRATIONS

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Let $\mathcal{A}$ be a (central) hyperplane arrangement in $\mathbb{C}^\ell$.

For each $H \in \mathcal{A}$, let $f_H: \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel $H$.

For each choice of multiplicities $m = (m_H)_{H \in \mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.

The map $Q_m: \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m: M(\mathcal{A}) \to \mathbb{C}^*$.

This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \xrightarrow{Q_m} M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$
Let \( \mathcal{A} \) be a (central) hyperplane arrangement in \( \mathbb{C}^\ell \).

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\[
F_m(\mathcal{A}) \xrightarrow{Q_m} M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.
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- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the Milnor fiber of the multi-arrangement.

- $F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $\ell - 1$.

- The (geometric) monodromy is the diffeomorphism
  \[
  h: F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i/N} z.
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- If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$. 
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**Example**

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = m$-roots of 1.

**Example**

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and $h$ is an automorphism of order 3:

More generally, if $\mathcal{A}$ is a pencil of $n$ lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\left(\frac{n-1}{2}\right)$, with $n$ punctures.
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Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential arrangement. The inclusion $\iota: M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m(\mathcal{A})} \mathbb{C}^*$$

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\end{array}$$

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Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$
Some basic questions about the topology of the Milnor fibration:

(Q1) Are the homology groups $H_q(F_m(A), \mathbb{k})$ determined by $L(A)$? If so, is the characteristic polynomial of the algebraic monodromy, $h_* : H_q(F_m(A), \mathbb{k}) \to H_q(F_m(A), \mathbb{k})$, also determined by $L(A)$?

(Q2) Are the homology groups $H_q(F_m(A), \mathbb{Z})$ torsion-free? If so, does $F_m(A)$ admit a minimal cell structure?

(Q3) Is $F_m(A)$ a (partially) formal space?
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Let \((\mathcal{A}, m)\) be a multi-arrangement with \(\gcd\{m_H | H \in \mathcal{A}\} = 1\). Set \(N = \sum_{H \in \mathcal{A}} m_H\).

The Milnor fiber \(F_m(\mathcal{A})\) is a regular \(\mathbb{Z}_N\)-cover of \(U(\mathcal{A}) = \mathbb{P}(\mathcal{M}(\mathcal{A}))\) defined by the homomorphism

\[
\delta_m : \pi_1(U(\mathcal{A})) \to \mathbb{Z}_N, \quad x_H \mapsto m_H \mod N
\]

Let \(\hat{\delta}_m : \text{Hom}(\mathbb{Z}_N, k^*) \to \text{Hom}(\pi_1(U(\mathcal{A})), k^*)\). If \(\text{char}(k) \nmid N\), then

\[
\dim_k H_q(F_m(\mathcal{A}), k) = \sum_{s \geq 1} \left| \mathcal{V}_s^q(U(\mathcal{A}), k) \cap \text{im}(\hat{\delta}_m) \right|
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This gives a formula for the polynomial \(\Delta_q(t) = \det(t \cdot \text{id} - h_*)\) in terms of the characteristic varieties of \(U(\mathcal{A})\).
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This gives a formula for the polynomial \(\Delta_q(t) = \det(t \cdot \text{id} - h_*)\) in terms of the characteristic varieties of \(U(\mathcal{A})\).
Write

$$\Delta(t) := \Delta_1(t) = \prod_{d|n} \Phi_d(t)^{e_d(A)},$$

where $\Phi_d(t)$ is the $d$-th cyclotomic polynomial, and $e_d(A) \in \mathbb{Z}_{\geq 0}$.

Transfer argument: $e_1(A) = n - 1$.

If there is a non-transverse multiple point on $A$ of multiplicity not divisible by $d$, then $e_d(A) = 0$. (Libgober 2002).

In particular, if $A$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t - 1)^{m-1}(t^2 + t + 1)^{e_3}$.

If multiplicity 4 appears, then also get factor of $(t + 1)^{e_2} \cdot (t^2 + 1)^{e_4}$.

**Example**

Let $A$ be the braid arrangement. $\mathcal{V}_1(A)$ has a single essential component, $T = \{t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 t_4 t_5 t_6^{-1} = t_2 t_5^{-1} = t_3 t_4^{-1} = 1\}$. Clearly, $\delta^2 \in T$, yet $\delta \notin T$; hence, $\Delta(t) = (t - 1)^5(t^2 + t + 1)$. 
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**Modular inequalities**

Let $\sigma = \sum_{H \in A} e_H \in A^1$ be the “diagonal” vector.

Assume $k$ has characteristic $p > 0$, and define

$$\beta_p(A) = \dim_k H^1(A, \cdot \sigma).$$

That is, $\beta_p(A) = \max \{ s \mid \sigma \in R_s^1(A, k) \}$.


$e_{ps}(A) \leq \beta_p(A)$, for all $s \geq 1$.

**Theorem**

1. Suppose $A$ admits a $k$-net. Then $\beta_p(A) = 0$ if $p \nmid k$ and $\beta_p(A) \geq k - 2$, otherwise.
2. If $A$ admits a reduced $k$-multinet, then $e_k(A) \geq k - 2$. 
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**Theorem**

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2. *If \( \mathcal{A} \) admits a reduced \( k \)-multinet, then \( e_k(\mathcal{A}) \geq k - 2 \).*
Suppose \( A \) has no points of multiplicity \( 3r \) with \( r > 1 \). Then \( A \) admits a reduced 3-multinet iff \( A \) admits a 3-net iff \( \beta_3(A) \neq 0 \). Moreover,

- \( \beta_3(A) \leq 2 \).
- \( e_3(A) = \beta_3(A) \), and thus \( e_3(A) \) is combinatorially determined.

Corollary (PS)

Suppose all flats \( X \in L_2(A) \) have multiplicity 2 or 3. Then \( \Delta(t) \), and thus \( b_1(F(A)) \), are combinatorially determined.

Theorem (PS)

Suppose \( A \) supports a 4-net and \( \beta_2(A) \leq 2 \). Then

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e_2(A) = e_4(A) = \beta_2(A) = 2.
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**Theorem (Papadima–S. 2014)**

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**Conjecture (PS)**

Let $\mathcal{A}$ be an arrangement which is not a pencil. Then $e_{ps}(\mathcal{A}) = 0$ for all primes $p$ and integers $s \geq 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors $d$ of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t - 1)^{|\mathcal{A}| - 1}((t + 1)(t^2 + 1))^{\beta_2(\mathcal{A})}(t^2 + t + 1)^{\beta_3(\mathcal{A})}.$$  

The conjecture has been verified for several classes of arrangements:

- Complex reflection arrangements (Măcinic–Papadima–Popescu).
- Certain types of real arrangements (Yoshinaga, Bailet, Torielli).
- Arrangements w/ connected multiplicity graph (Salvetti–Serventi).
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For every prime $p \geq 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion.

Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2 y(x^2 - y^2)^3(x^2 - z^2)^2(y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 
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We now can generalize and reinterpret these examples, as follows.

A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

**Theorem (Denham–S. 2014)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1^1(M(\mathcal{A}'), \mathbb{k})$ varies with $\text{char}(\mathbb{k})$. 
We now can generalize and reinterpret these examples, as follows.

A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H | n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

**Theorem (Denham–S. 2014)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\setminus\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

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To produce $p$-torsion in the homology of the usual Milnor fiber, we use a “polarization" construction:

$\{A, m\} \sim A \uparrow m$, an arrangement of $N = \sum_{H \in A} m_H$ hyperplanes, of rank equal to $\text{rank } A + |\{H \in A: m_H \geq 2\}|$.

**Theorem (DS)**

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There is then a choice of multiplicities $m'$ on the deletion $A' = A \setminus \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has $p$-torsion, where $\mathcal{B} = A' \uparrow m'$ and $q = 1 + |\{K \in A': m'_K \geq 3\}|$. 
To produce $p$-torsion in the homology of the usual Milnor fiber, we use a “polarization” construction:

$(\mathcal{A}, m) \sim \mathcal{A} \parallel m$, an arrangement of $N = \sum_{H \in \mathcal{A}} m_H$ hyperplanes, of rank equal to $\text{rank } \mathcal{A} + |\{H \in \mathcal{A}: m_H \geq 2\}|$.

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**Corollary (DS)**

*For every prime $p \geq 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion, for some $q > 1$.***

Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1w_2w_3w_4w_5(x^2 - w_1^2)(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1) \cdot ((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)((x - z)^2 - w_4^2)((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).
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Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).
**Example (Zuber 2010)**

- Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^3$ defined by
  \[ Q = (z_1^3 - z_2^3)(z_1^3 - z_3^3)(z_2^3 - z_3^3). \]
- The variety $\mathcal{R}^1(M) \subset \mathbb{C}^9$ has 12 local components (from triple points), and 4 essential components (from 3-nets).
- One of these 3-nets corresponds to the rational map $\mathbb{C}P^2 \dashrightarrow \mathbb{C}P^1$, $(z_1, z_2, z_3) \mapsto (z_1^3 - z_2^3, z_2^3 - z_3^3)$.
- This map can be used to construct a 4-dimensional subtorus $T = \exp(L)$ inside $\text{Hom}(\pi_1(F(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^{12}$.
- The subspace $L \subset \mathcal{H}^1(F(\mathcal{A}), \mathbb{C})$ is not a component of $\mathcal{R}^1(F(\mathcal{A}))$.
- Thus, the tangent cone formula is violated, and so the Milnor fiber $F(\mathcal{A})$ is not 1-formal.
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The formality problem

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