Hyperplane arrangements: at the crossroads of topology and combinatorics

Alex Suciu
Northeastern University

Colloquium
Goethe University Frankfurt
October 25, 2013
An arrangement of hyperplanes is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.

Intersection lattice $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.

Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$.

The Boolean arrangement $\mathcal{B}_n$
- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type $A_{n-1}$)
- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1, \ldots, n\}$.
- $M(\mathcal{A}_n)$: configuration space of $n$ ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on $n$ strings).
Figure: A planar slice of the braid arrangement $\mathcal{A}_4$
We may assume that $A$ is essential, i.e., $\bigcap_{H \in A} H = \{0\}$.

Fix an ordering $A = \{H_1, \ldots, H_n\}$, and choose linear forms $f_i: \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$.

Define an injective linear map

$$\iota_A: \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \ldots, f_n(z)).$$

This map restricts to an inclusion $\iota: M(A) \hookrightarrow M(B_n)$. Thus,

$$M(A) = \iota_A(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n,$$

a "very affine" subvariety of $(\mathbb{C}^*)^n$.

The tropicalization of this sub variety is a fan in $\mathbb{R}^n$. Feichtner and Sturmfels: this is the Bergman fan of $L(A)$. 
• \( M(\mathcal{A}) \) has the homotopy type of a connected, finite cell complex of dimension \( \ell \).

• In fact, \( M = M(\mathcal{A}) \) admits a minimal cell structure. Consequently, \( H_*(M, \mathbb{Z}) \) is torsion-free.

• The Betti numbers \( b_q(M) := \text{rank } H_q(M, \mathbb{Z}) \) are given by

\[
\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X) (-t)^{\text{rank}(X)},
\]

where \( \mu : L(\mathcal{A}) \to \mathbb{Z} \) is the Möbius function, defined recursively by \( \mu(\mathcal{C}^\ell) = 1 \) and \( \mu(X) = -\sum_{Y \supseteq X} \mu(Y) \).

• The Orlik–Solomon algebra \( H^*(M, \mathbb{Z}) \) is the quotient of the exterior algebra on generators \( \{e_H \mid H \in \mathcal{A}\} \) by an ideal determined by the circuits in the matroid of \( \mathcal{A} \).

• Thus, the ring \( H^*(M, \mathbb{K}) \) is determined by \( L(\mathcal{A}) \), for every field \( \mathbb{K} \).
Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\overline{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.

$L_1(\mathcal{A}) \leftrightarrow$ lines of $\overline{\mathcal{A}}$, $L_2(\mathcal{A}) \leftrightarrow$ intersection points of $\overline{\mathcal{A}}$, poset structure of $L_{\leq 2}(\mathcal{A}) \leftrightarrow$ incidence structure of $\overline{\mathcal{A}}$.

A flat $X \in L_2(\mathcal{A})$ has multiplicity $q$ if the point $\overline{X}$ has exactly $q$ lines from $\overline{\mathcal{A}}$ passing through it.

**Definition (Falk and Yuzvinsky)**

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geq 3$ subsets $\mathcal{A}_1, \ldots, \mathcal{A}_k$, together with an assignment of multiplicities, $m: \mathcal{A} \rightarrow \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

1. There is an integer $d$ such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If $H$ and $H'$ are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha: H \supseteq X} m_H$ is independent of $\alpha$.
4. Each $(\bigcup_{H \in \mathcal{A}_\alpha} H) \setminus \mathcal{X}$ is connected.
A multinet as above is also called a \((k, d)\)-multinet, or a \(k\)-multinet.

If \(m_H = 1\), for all \(H \in \mathcal{A}\), the multinet is \textit{reduced}.

If, furthermore, \(n_X = 1\), for all \(X \in \mathcal{X}\), this is a \textit{net}. In this case, \(|\mathcal{A}_\alpha| = |\mathcal{A}| / k = d\), for all \(\alpha\). Moreover, \(\bar{\mathcal{X}}\) has size \(d^2\), and is encoded by a \((k - 2)\)-tuple of orthogonal Latin squares.

A \((3, 2)\)-net on the \(A_3\) arrangement \(\mathcal{A}\) consists of 4 triple points \((n_X = 1)\)

A \((3, 4)\)-multinet on the \(B_3\) arrangement \(\mathcal{B}\) consists of 4 triple points \((n_X = 1)\) and 3 triple points \((n_X = 2)\)
A $(3, 3)$-net on the Ceva matroid. A $(4, 3)$-net on the Hessian matroid.

- If $\mathcal{A}$ has no flats of multiplicity $kr$, for some $r > 1$, then every reduced $k$-multinet is a $k$-net.
- (Yuzvinsky and Pereira–Yuzvinsky): If $\mathcal{A}$ supports a $k$-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; moreover, if the multinet is not reduced, then $k = 3$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4, 3)$-net.
Cohomology jump loci

Let $X$ be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

Let $k$ be an algebraically closed field, and let $\text{Hom}(\pi, k^*)$ be the affine algebraic group of $k$-valued, multiplicative characters on $\pi$.

The characteristic varieties of $X$ are the jump loci for homology with coefficients in rank-1 local systems on $X$:

$$V_s^q(X, k) = \{ \rho \in \text{Hom}(\pi, k^*) \mid \dim_k H_q(X, k\rho) \geq s \}.$$  

Here, $k\rho$ is the local system defined by $\rho$, i.e., $k$ viewed as a $k\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, k\rho) = H_i(C_*(\tilde{X}, k) \otimes_{k\pi} k\rho)$.

These loci are Zariski closed subsets of the character group.

The sets $V_s^1(X, k)$ depend only on $\pi/\pi''$. 
**Example (Circle)**

We have $\widetilde{S}^1 = \mathbb{R}$. Identify $\pi_1(S^1, \ast) = \mathbb{Z} = \langle t \rangle$ and $k\mathbb{Z} = k[t^{\pm 1}]$.  

Then:  

$$C_\ast(\widetilde{S}^1, k) : 0 \rightarrow k[t^{\pm 1}] \overset{t^{-1}}{\rightarrow} k[t^{\pm 1}] \rightarrow 0.$$  

For $\rho \in \text{Hom}(\mathbb{Z}, k^*) = k^*$, we get  

$$C_\ast(\widetilde{S}^1, k) \otimes_{k\mathbb{Z}} k_\rho : 0 \rightarrow k \overset{\rho^{-1}}{\rightarrow} k \rightarrow 0,$$  

which is exact, except for $\rho = 1$, when $H_0(S^1, k) = H_1(S^1, k) = k$.  

Hence: $\mathcal{V}^0_1(S^1, k) = \mathcal{V}^1_1(S^1, k) = \{1\}$ and $\mathcal{V}^i_s(S^1, k) = \emptyset$, otherwise.

**Example (Punctured complex line)**

Identify $\pi_1(\mathbb{C}\setminus\{n \text{ points}\}) = F_n$, and $\hat{F}_n = (k^*)^n$. Then:  

$$\mathcal{V}^1_s(\mathbb{C}\setminus\{n \text{ points}\}, k) = \begin{cases} (k^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$
Let $A = H^*(X, \mathbb{k})$. If $\text{char } \mathbb{k} = 2$, assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$.

Thus, we get a cochain complex

$$(A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots,$$

known as the Aomoto complex of $A$.

The resonance varieties of $X$ are the jump loci for the Aomoto-Betti numbers

$$\mathcal{R}_s^q(X, \mathbb{k}) = \{ a \in A^1 \mid \dim_{\mathbb{k}} H^q(A, \cdot a) \geq s \},$$

These loci are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{k})$.

**Example**

- $\mathcal{R}_1^1(T^n, \mathbb{k}) = \{0\}$, for all $n > 0$.
- $\mathcal{R}_1^1(C \setminus \{n \text{ points}\}, \mathbb{k}) = \mathbb{k}^n$, for all $n > 1$.
- $\mathcal{R}_1^1(\Sigma g, \mathbb{k}) = \mathbb{k}^{2g}$, for all $g > 1$.
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^3$, and identify $H^1(M(\mathcal{A}), \mathbb{k}) = \mathbb{k}^n$, with basis dual to the meridians.

The resonance varieties $\mathcal{R}_s^1(\mathcal{A}, \mathbb{k}) := \mathcal{R}_s^1(M(\mathcal{A}), \mathbb{k}) \subset \mathbb{k}^n$ lie in the hyperplane $\{x \in \mathbb{k}^n \mid x_1 + \cdots + x_n = 0\}$.

$\mathcal{R}(\mathcal{A}) = \mathcal{R}_1^1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $\mathbb{C}^n$.

Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.

$\mathcal{R}_s^1(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$. 
Each flat \(X \in L_2(\mathcal{A})\) of multiplicity \(k \geq 3\) gives rise to a local component of \(R(\mathcal{A})\), of dimension \(k - 1\).

More generally, every \(k\)-multinet of a sub-arrangement \(B \subseteq \mathcal{A}\) gives rise to a component of dimension \(k - 1\), and all components of \(R(\mathcal{A})\) arise in this way.

The resonance varieties \(R^1(\mathcal{A}, k)\) can be more complicated, e.g., they may have non-linear components.
**Example (Braid arrangement $A_4$)**

$R^1(A, \mathbb{C}) \subset \mathbb{C}^6$ has 4 local components (from triple points), and one non-local component, from the $(3, 2)$-net:

$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$

$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$

$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$

$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$

$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$
• Let \( \text{Hom}(\pi_1(M), \mathbb{k}^*) = (\mathbb{k}^*)^n \) be the character torus.

• The characteristic variety \( \mathcal{V}^1(\mathcal{A}, \mathbb{k}) := \mathcal{V}^1_1(M(\mathcal{A}), \mathbb{k}) \subset (\mathbb{k}^*)^n \) lies in the substorus \( \{ t \in (\mathbb{k}^*)^n \mid t_1 \cdots t_n = 1 \} \).

• \( \mathcal{V}^1(\mathcal{A}, \mathbb{C}) \) is a finite union of torsion-translates of algebraic subtori of \( (\mathbb{C}^*)^n \).

• If a linear subspace \( L \subset \mathbb{C}^n \) is a component of \( \mathcal{R}^1(\mathcal{A}, \mathbb{C}) \), then the algebraic torus \( T = \exp(L) \) is a component of \( \mathcal{V}^1(\mathcal{A}, \mathbb{C}) \).

• All components of \( \mathcal{V}^1(\mathcal{A}, \mathbb{C}) \) passing through the origin \( 1 \in (\mathbb{C}^*)^n \) arise in this way (and thus, are combinatorially determined).
In general, though, there are translated subtori in $V^1(\mathcal{A}, k)$.

When this happens, the characteristic varieties $V^1(\mathcal{A}, k)$ may depend (qualitatively) on $\text{char}(k)$. 
The Milnor fibration(s) of an arrangement

For each $H \in \mathcal{A}$, let $f_H : \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel $H$.

For each choice of multiplicities $m = (m_H)_{H \in \mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H},$$

a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.

The map $Q_m : \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m : M(\mathcal{A}) \to \mathbb{C}^*.$

This is the projection of a smooth, locally trivial bundle, known as the Milnor fibration of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \xrightarrow{\quad} M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$
The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.

$F_m(\mathcal{A})$ has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, and of dimension $\ell - 1$.

The *(geometric) monodromy* is the diffeomorphism

$$h: F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i/N}z.$$

If all $m_H = 1$, the polynomial $Q = Q_m(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

**Example**

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then:

- $M(\mathcal{A}) = \mathbb{C}^*$.
- $Q_m(\mathcal{A}) = z^m$.
- $F_m(\mathcal{A}) = m$-roots of 1.
**Example**

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and $h$ is an automorphism of order 3:

![Diagram](image)

More generally, if $\mathcal{A}$ is a pencil of $n$ lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with $n$ punctures.
Let $B_n$ be the Boolean arrangement, with $Q_m(B_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(B_n) = (\mathbb{C}^*)^n$ and

$$F_m(B_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

Let $A = \{H_1, \ldots, H_n\}$ be an essential arrangement. The inclusion $\iota_A : M(A) \to M(B_n)$ restricts to a bundle map

$$
\begin{align*}
F_m(A) & \to M(A) \xrightarrow{Q_m(A)} \mathbb{C}^* \\
F_m(B_n) & \to M(B_n) \xrightarrow{Q_m(B_n)} \mathbb{C}^*
\end{align*}
$$

Thus,

$$F_m(A) = M(A) \cap F_m(B_n)$$

The tropicalization of $F_m(A)$ is a fan in $\mathbb{R}^{n-1}$. Question: Is this fan determined by $L(A)$ (and the multiplicity vector $m$)?
Two basic questions about the topology of the Milnor fibration(s):

(Q1) Are the homology groups \( H_q(F(\mathcal{A}), \mathbb{C}) \) determined by \( L(\mathcal{A}) \)?
If so, is the characteristic polynomial of the algebraic monodromy, \( h_* : H_q(F(\mathcal{A}), \mathbb{C}) \to H_q(F(\mathcal{A}), \mathbb{C}) \), also determined by \( L(\mathcal{A}) \)?

(Q2) Are the homology groups \( H_q(F(\mathcal{A}), \mathbb{Z}) \) torsion-free?
If so, does \( F(\mathcal{A}) \) admit a minimal cell structure?

Some recent progress on these questions:

- A partial, positive answer to (Q1): joint work with Stefan Papadima (in progress).
- A negative answer to (Q2): joint work with Graham Denham (to appear).
Let \((A, m)\) be a multi-arrangement with \(\gcd\{m_H \mid H \in A\} = 1\). Set \(N = \sum_{H \in A} m_H\).

The Milnor fiber \(F_m(A)\) is a regular \(\mathbb{Z}_N\)-cover of \(U = \mathbb{P}(M(A))\) defined by the homomorphism \(\delta_m: \pi_1(U) \to \mathbb{Z}_N, \ x_H \mapsto m_H \mod N\).

Let \(\hat{\delta}_m: \text{Hom}(\mathbb{Z}_N, \mathbb{k}^*) \to \text{Hom}(\pi_1(U), \mathbb{k}^*)\). If \(\text{char}(\mathbb{k}) \nmid N\), then

\[
\dim_{\mathbb{k}} H_q(F_m(A), \mathbb{k}) = \sum_{s \geq 1} \left| V_s^q(U, \mathbb{k}) \cap \text{im}(\hat{\delta}_m) \right|.
\]
**Multinets and $H_1(F(A), \mathbb{C})$**

- Recall: the monodromy $h: F(A) \to F(A)$ has order $n = |A|$.
- Thus, the characteristic polynomial of $h_\ast$ acting on $H_1(F(A), \mathbb{C})$ can be written as

$$\Delta(t) := \det(h_\ast - t \cdot \text{id}) = \prod_{d|n} \Phi_d(t)^{e_d(A)},$$

where $\Phi_1 = t - 1$, $\Phi_2 = t + 1$, $\Phi_3 = t^2 + t + 1$, ... are the cyclotomic polynomials, and $e_d(A) \in \mathbb{Z}_{\geq 0}$.

- Easy to see: $e_1(A) = n - 1$. Thus, for $q = 1$, question (Q1) is equivalent to: are the integers $e_d(A)$ determined by $L_{\leq 2}(A)$?

**Proposition**

*If $\mathcal{A}$ admits a reduced $k$-multinet, then $e_k(A) \geq k - 2$.***
Let $A^* = H^*(M(A), k)$, where $k$ is a field of characteristic $p > 0$.

Let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$ be the "diagonal" vector.

Define the mod-$p$ Aomoto-Betti number of $\mathcal{A}$ as

$$\beta_p(\mathcal{A}) = \dim_k H^1(\mathcal{A}, \cdot \sigma).$$

$\beta_p(\mathcal{A})$ depends only on $L(\mathcal{A})$ and $p$, and $0 \leq \beta_p(\mathcal{A}) \leq |\mathcal{A}| - 2$.

(Cohen–Orlik 2000, Papadima–S. 2010) $e_{p^s}(\mathcal{A}) \leq \beta_p(\mathcal{A})$.

**Theorem (Papadima–S. 2013)**

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, for some $r > 1$. Then $\beta_3(\mathcal{A}) \leq 2$.

Moreover, $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and so $e_3(\mathcal{A})$ is combinatorially determined.

A similar result holds for $e_2(\mathcal{A})$. 
**Lemma (PS)**

If $\mathcal{A}$ supports a 3-net with parts $\mathcal{A}_\alpha$, then:

1. $1 \leq \beta_3(\mathcal{A}) \leq \beta_3(\mathcal{A}_\alpha) + 1$, for all $\alpha$.
2. If $\beta_3(\mathcal{A}_\alpha) = 0$, for some $\alpha$, then $\beta_3(\mathcal{A}) = 1$.
3. If $\beta_3(\mathcal{A}_\alpha) = 1$, for some $\alpha$, then $\beta_3(\mathcal{A}) = 1$ or 2.

All possibilities do occur:

- **Braid arrangement:** has a $(3, 2)$-net from the Latin square of $\mathbb{Z}_2$.
  
  $\beta_3(\mathcal{A}_\alpha) = 0$ (\forall \alpha) and $\beta_3(\mathcal{A}) = 1$.

- **Pappus arrangement:** has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$.
  
  $\beta_3(\mathcal{A}_1) = \beta_3(\mathcal{A}_2) = 0$, $\beta_3(\mathcal{A}_3) = 1$ and $\beta_3(\mathcal{A}) = 1$.

- **Ceva arrangement:** has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$.
  
  $\beta_3(\mathcal{A}_\alpha) = 1$ (\forall \alpha) and $\beta_3(\mathcal{A}) = 2$. 

**Theorem (PS)**

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, for some $r > 1$. Then $\beta_3(\mathcal{A}) \leq 2$. Moreover, the following conditions are equivalent:

1. $\mathcal{A}$ admits a reduced $3$-multinet.
2. $\mathcal{A}$ admits a $3$-net.
3. $\beta_3(\mathcal{A}) \neq 0$.

**Remark**

- One may define $\beta_p(\mathcal{M})$ for any matroid $\mathcal{M}$.
- For each $n \in \mathbb{N}$, there exists a matroid $\mathcal{M}_n$ supporting a $(3, 3^n)$-net corresponding to $\mathbb{Z}_3^n$, such that $\beta_3(\mathcal{M}_n) = n + 1$.
- By the above, such a matroid is realizable by an arrangement in $\mathbb{C}^3$ if and only if $n = 1$. 
**Theorem (Cohen–Denham–S. 2003)**

For every prime $p \geq 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion.

Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2 y (x^2 - y^2)^3 (x^2 - z^2)^2 (y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. 
We now can generalize and reinterpret these examples, as follows.

A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H | n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

**Theorem (Denham–S. 2013)**

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane $H$ and multiplicity $m$. Let $p$ be a prime dividing $m_H$. There is then a choice of multiplicities $m'$ on the deletion $\mathcal{A}' = \mathcal{A}\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero $p$-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}(\mathcal{A}', \mathbb{k})$ varies with $\text{char}(\mathbb{k})$. 
To produce \( p \)-torsion in the homology of the usual Milnor fiber, we use a “polarization” construction:

\[
(A, m) \leadsto A \parallel m, \text{ an arrangement of } N = \sum_{H \in A} m_H \text{ hyperplanes, of rank equal to } \text{rank } A + |\{H \in A : m_H \geq 2\}|.
\]

**Theorem (DS)**

Suppose \( A \) admits a pointed multinet, with distinguished hyperplane \( H \) and multiplicity \( m \). Let \( p \) be a prime dividing \( m_H \).

There is then a choice of multiplicities \( m' \) on the deletion \( A' = A \setminus \{H\} \) such that \( H_q(F(B), \mathbb{Z}) \) has \( p \)-torsion, where \( B = A' \parallel m' \) and \( q = 1 + |\{K \in A' : m'_K \geq 3\}|. \)
**Corollary (DS)**

For every prime $p \geq 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero $p$-torsion, for some $q > 1$.

Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1w_2w_3w_4w_5(x^2 - w_1^2)(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1).$$

$$((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)((x - z)^2 - w_4^2)((x - z)^2 - 2w_4^2) \cdot ((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).
REFERENCES

