Northeastern University
Department of Mathematics

# Qualifying Exam in Topology

## September 2007

*Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as* **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $X$ and $Y$ be compact, Hausdorff spaces. Let $A \subset X$ and $B \subset Y$ be closed subspaces. Suppose $U$ is an open subset in the product space $X \times Y$, and $U$ contains $A \times B$. Show that there exist open subsets $V \subset X$ and $W \subset Y$ such that $A \times B \subset V \times W \subset U$.

2. Let $U$ be an open subset of $\mathbb{R}^n$. Show that $U$ is connected if and only if $U$ is path-connected.

3. A subspace $A \subset X$ is called a *retract* if there is a continuous map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$, where $i\colon A \to X$ denotes the inclusion. Prove or disprove the following:
   (a) The sphere $S^2$ retracts onto its equator $S^1$.
   (b) The real line $\mathbb{R}$ retracts onto the unit interval $I = [0, 1]$.
   (c) The Möbius band retracts onto its boundary circle.

4. Let $K$ be the Klein bottle—a square with opposite vertical edges identified in the same direction, and opposite horizontal edges identified in the opposite direction.
   (a) Describe all 2-fold and 3-fold connected covering spaces of $K$, up to equivalence.
   (b) For each covering, indicate the corresponding subgroup of $\pi_1(K)$.

5. Let $A$ be the torus with an open 2-disk removed, and let $B$ be another copy of the 2-disk. Let $f\colon \partial B \to \partial A$ be the map winding twice (upon identifying $\partial A$ and $\partial B$ with the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the map $f$ is given by $f(z) = z^2$). Finally, let $X = A \cup_f B$ be the space obtained from $A$ by adjoining the 2-cell $B$ along the attaching map $f$.
   (a) Compute the fundamental group of $X$.
   (b) Compute all the homology groups of $X$.

6. Answer the questions below. Justify your answers by either (i) drawing or otherwise describing the cover if there is one, or (ii) proving that no such cover exists if there is none.
   (a) Does the connected sum of 2 tori cover the connected sum of 3 projective planes?
   (b) Does the connected sum of 5 tori cover the connected sum of 4 tori?
   (c) Does the connected sum of 5 tori cover the connected sum of 3 tori?
   (d) Does the connected sum of 4 projective planes cover the connected sum of 2 tori?