Northeastern University
Department of Mathematics

Qualifying Exam in Topology

Spring 1998

Do six of the following seven problems. Give complete proofs or justifications for each statement you make. Show all your work.

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1. Let $\{X_n\}_{n=1}^{\infty}$ be a countable family of locally compact topological spaces. Show that the product space $\prod_{n=1}^{\infty} X_n$ is locally compact if and only if all but finitely many of the spaces X_n are compact.

2. Consider the following two topologies on $\mathbb{R}^2$:
   (i) The Zariski topology, $\mathcal{T}_1$, where a base of open sets is formed by the complements of zero sets of polynomials in two variables.
   (ii) The topology $\mathcal{T}_2$, the weakest topology where all straight lines are closed sets.

   Show that $(\mathbb{R}^2, \mathcal{T}_1)$ and $(\mathbb{R}^2, \mathcal{T}_2)$ are not homeomorphic.

3. Let $X = S^n/\{p_1, \ldots, p_k\}$ be the topological space obtained from the n-sphere by identifying k distinct points on it, $k \geq 2$.
   (a) Find the fundamental group of X.
   (b) Find the homology groups of X.

4. Consider two labeled dodecagons with edges identified in pairs, as follows:
   (i) $abcda^{-1}b^{-1}c^{-1}d^{-1}efe^{-1}f^{-1}$.
   (ii) $abcda^{-1}b^{-1}c^{-1}defe^{-1}f^{-1}$.

   For each of the resulting surfaces, determine its orientability status (i.e., orientable or not), its genus (i.e., number of handles or crosscaps), and its Euler characteristic.

5. Let K be the Klein bottle—a square with opposite vertical edges identified in the same direction, and opposite horizontal edges identified in the opposite direction.
   (a) Describe all covering spaces of K, up to equivalence.
   (b) For each covering, indicate the corresponding subgroup of $\pi_1(K)$.

6. Let S^2 be the 2-sphere, and T^2 be the 2-torus. Prove the following:
   (a) Every continuous map $S^2 \to T^2$ is homotopic to a constant map.
   (b) There exists a continuous map $T^2 \to S^2$ which is not homotopic to a constant map.

7. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^3$, where $\mathbb{RP}^n$ is the n-dimensional real projective space.
   (a) Find a CW-decomposition of X.
   (b) Determine the chain complex $(C_\bullet(X), d)$ associated to that cell decomposition.
   (c) Compute the homology groups $H_*(X)$.