Northeastern University
Department of Mathematics

Qualifying Exam in Topology

April 2007

Do the following five problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, and let $f\colon X \to \mathbb{R}$ be the function defined by $f(x, y) = x$. Prove or disprove the following:
   (a) f is a continuous map.
   (b) f is an open map.
   (c) f is a closed map.
   (d) f is a quotient map.
   (e) f is a local homeomorphism.

2. Prove or disprove the following:
   (a) If X and Y are path-connected, then $X \times Y$ is path-connected.
   (b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.
   (c) If X is locally path-connected, and $A \subset X$, then A is locally path-connected.
   (d) If X is path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.
   (e) If X is locally path-connected, and $f\colon X \to Y$ is continuous, then $f(X)$ is locally path-connected.

3. Let $X = S^1 \bigvee S^1$ be the wedge of two circles at the basepoint x_0.
   (a) Explain why we may identify $\pi_1(X, x_0)$ with F_2, the free group of rank 2.
   (b) Consider the homomorphism $\phi\colon F_2 \to S_3$ that maps the first generator to the transposition $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$, and the second generator to the 3-cycle $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$. Draw the covering space $p\colon Y \to X$ whose associated lifting correspondence is ϕ.
   (c) Is p a normal (that is, regular) cover?
   (d) Fix a basepoint $y_0 \in p^{-1}(x_0)$ and compute $\pi_1(Y, y_0)$.
   (e) Determine the homomorphim $p_\sharp\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$.
   (f) Determine the homomorphim $p_*\colon H_1(Y) \to H_1(X)$.
   (g) Does the index of $p_\sharp(\pi_1(Y, y_0))$ in $\pi_1(X, x_0)$ coincide with the index of $p_*(H_1(Y))$ in $H_1(X)$?

4. Let $p\colon E \to B$ be a covering space. Fix a basepoint $b_0 \in B$, and suppose $p^{-1}(b_0)$ has k elements.
   (a) Assume B is connected. Show that $p^{-1}(b)$ has also k elements, for any $b \in B$.
   (b) Assume that, in addition, B is compact. Show that E is also compact.

5. Let S^2 be the 2-sphere, $\mathbb{RP}^2$ the projective plane, and T^2 the 2-torus.
   (a) For each of these 3 spaces, compute the fundamental group, and identify the universal cover.
   (b) Prove or disprove the following:
      (i) Every continuous map $S^2 \to \mathbb{RP}^2$ is homotopic to a constant map.
      (ii) Every continuous map $S^2 \to T^2$ is homotopic to a constant map.
      (iii) Every continuous map $\mathbb{RP}^2 \to S^2$ is homotopic to a constant map.
      (iv) Every continuous map $\mathbb{RP}^2 \to T^2$ is homotopic to a constant map.
      (v) Every continuous map $T^2 \to S^2$ is homotopic to a constant map.
      (vi) Every continuous map $T^2 \to \mathbb{RP}^2$ is homotopic to a constant map.