1. Let $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, and let $f : X \to \mathbb{R}$ be the function defined by $f(x, y) = x$. Prove or disprove the following:
   (a) $f$ is a continuous map.
   (b) $f$ is an open map.
   (c) $f$ is a closed map.
   (d) $f$ is a quotient map.
   (e) $f$ is a local homeomorphism.

2. Prove or disprove the following:
   (a) If $X$ and $Y$ are path-connected, then $X \times Y$ is path-connected.
   (b) If $A \subset X$ is path-connected, then $\overline{A}$ is path-connected.
   (c) If $X$ is locally path-connected, and $A \subset X$, then $A$ is locally path-connected.
   (d) If $X$ is path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.
   (e) If $X$ is locally path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is locally path-connected.

3. Let $X = S^1 \vee S^1$ be the wedge of two circles at the basepoint $x_0$.
   (a) Explain why we may identify $\pi_1(X, x_0)$ with $F_2$, the free group of rank 2.
   (b) Consider the homomorphism $\phi : F_2 \to S_3$ that maps the first generator to the transposition $(1 \ 2 \ 3)$, and the second generator to the 3-cycle $(1 \ 3 \ 2)$. Draw the covering space $p : Y \to X$ whose associated lifting correspondence is $\phi$.
   (c) Is $p$ a normal (that is, regular) cover?
   (d) Fix a basepoint $y_0 \in p^{-1}(x_0)$ and compute $\pi_1(Y, y_0)$.
   (e) Determine the homomorphism $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.
   (f) Determine the homomorphism $p_* : H_1(Y) \to H_1(X)$.
   (g) Does the index of $p_* (\pi_1(Y, y_0))$ in $\pi_1(X, x_0)$ coincide with the index of $p_* (H_1(Y))$ in $H_1(X)$?
4. Let $p: E \to B$ be a covering space. Fix a basepoint $b_0 \in B$, and suppose $p^{-1}(b_0)$ has $k$ elements.
   (a) Assume $B$ is connected. Show that $p^{-1}(b)$ has also $k$ elements, for any $b \in B$.
   (b) Assume that, in addition, $B$ is compact. Show that $E$ is also compact.

5. Let $S^2$ be the 2-sphere, $\mathbb{RP}^2$ the projective plane, and $T^2$ the 2-torus.
   (a) For each of these 3 spaces, compute the fundamental group, and identify the universal cover.
   (b) Prove or disprove the following:
      (i) Every continuous map $S^2 \to \mathbb{RP}^2$ is homotopic to a constant map.
      (ii) Every continuous map $S^2 \to T^2$ is homotopic to a constant map.
      (iii) Every continuous map $\mathbb{RP}^2 \to S^2$ is homotopic to a constant map.
      (iv) Every continuous map $\mathbb{RP}^2 \to T^2$ is homotopic to a constant map.
      (v) Every continuous map $T^2 \to S^2$ is homotopic to a constant map.
      (vi) Every continuous map $T^2 \to \mathbb{RP}^2$ is homotopic to a constant map.