Do six of the following seven problems. Give proofs or justifications for each statement you make. Be as clear and concise as possible. Show all your work.

1. Let \( f: X \to Y \) be a continuous map between topological spaces. Assume that \( Y \) is a Hausdorff space. Show that the graph \( \Gamma_f = \{ (x, y) \in X \times Y \mid y = f(x) \} \) is closed in \( X \times Y \).

2. Let \( X = \{ (x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q} \} \) be the set of all points in the plane with at least one rational coordinate. Show that \( X \), with the induced topology, is a connected space.

3. For each pair of spaces \((X, A)\) below, determine whether a retraction \( r: X \to A \) exists. If it does, sketch a construction of such a retraction; if it doesn’t, explain why not.
   (a) \( X = \mathbb{R}, \) with \( A = [0, 1] \).
   (b) \( X = \mathbb{R}, \) with \( A = (0, 1) \).
   (c) \( X \) the disk \( D^2 \), with \( A \) its boundary circle.
   (d) \( X \) the Möbius band, with \( A \) its boundary circle.

4. Let \( X = \mathbb{R}^3 \setminus \{ \{x\text{-axis}\} \cup \{y\text{-axis}\} \cup \{z\text{-axis}\} \} \). Compute the fundamental group of \( X \).

5. Let \( M_g \) be the compact, connected, orientable surface of genus \( g \). Prove the following.
   (a) If \( M_g \) is a covering of \( M_h \), then \( g = n(h - 1) + 1 \), for some \( n \) (equal to the number of sheets).
   (b) Conversely, if \( g = n(h - 1) + 1 \), there is an \( n \)-fold covering \( M_g \to M_h \).

6. Let \( \mathbb{R}P^2 \) be the (real) projective plane. Find all the connected covering spaces of
   (a) \( \mathbb{R}P^2 \vee \mathbb{R}P^2 \), the one-point union of two copies of \( \mathbb{R}P^2 \);
   (b) \( \mathbb{R}P^2 \sharp \mathbb{R}P^2 \), the connected sum of two copies of \( \mathbb{R}P^2 \) (also known as the Klein bottle).

7. Let \( X = S^1 \times K \) be the product of the circle with the Klein bottle.
   (a) Find a CW-decomposition of \( X \).
   (b) Determine the chain complex \( (C_*(X), d) \) associated to that cell decomposition.
   (c) Compute the homology groups \( H_*(X) \).