Northeastern University
Department of Mathematics

# Qualifying Exam in Topology

## Spring 2003

*Do six of the following seven problems. Give proofs or justifications for each statement you make. Be as clear and concise as possible. Show all your work.*

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1. Let $f\colon X \to Y$ be a continuous map between topological spaces. Assume that $Y$ is a Hausdorff space. Show that the graph $\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

2. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ be the set of all points in the plane with at least one rational coordinate. Show that $X$, with the induced topology, is a connected space.

3. For each pair of spaces $(X, A)$ below, determine whether a retraction $r\colon X \to A$ exists. If it does, sketch a construction of such a retraction; if it doesn't, explain why not.
   (a) $X = \mathbb{R}$, with $A = [0, 1]$.
   (b) $X = \mathbb{R}$, with $A = (0, 1)$.
   (c) $X$ the disk $D^2$, with $A$ its boundary circle.
   (d) $X$ the Möbius band, with $A$ its boundary circle.

4. Let $X = \mathbb{R}^3 \setminus (\{x\text{-axis}\} \cup \{y\text{-axis}\} \cup \{z\text{-axis}\})$. Compute the fundamental group of $X$.

5. Let $M_g$ be the compact, connected, orientable surface of genus $g$. Prove the following.
   (a) If $M_g$ is a covering of $M_h$, then $g = n(h - 1) + 1$, for some $n$ (equal to the number of sheets).
   (b) Conversely, if $g = n(h - 1) + 1$, there is an $n$-fold covering $M_g \to M_h$.

6. Let $\mathbb{RP}^2$ be the (real) projective plane. Find all the connected covering spaces of
   (a) $\mathbb{RP}^2 \vee \mathbb{RP}^2$, the one-point union of two copies of $\mathbb{RP}^2$;
   (b) $\mathbb{RP}^2 \sharp \mathbb{RP}^2$, the connected sum of two copies of $\mathbb{RP}^2$ (also known as the Klein bottle).

7. Let $X = S^1 \times K$ be the product of the circle with the Klein bottle.
   (a) Find a CW-decomposition of $X$.
   (b) Determine the chain complex $(C_\bullet(X), d)$ associated to that cell decomposition.
   (c) Compute the homology groups $H_*(X)$.