1. Let $M$ be an $n$-dimensional manifold. Let $\omega \in \Omega^n(M)$ be an $n$-form on $M$, and let $X$ be a complete vector field on $M$, with flow $\phi_t$. Prove that $\phi_t^* \omega = \omega$ if and only if $i_X \omega$ is closed.

2. Let $M$ be a compact orientable manifold of dimension $n$. Let $\alpha \in \Omega^n(M)$ be an $n$-form on $M$ and $X$ a vector field on $M$. Show that $L_X \alpha$ vanishes at some point.

3. Let $M$ and $N$ be smooth manifolds, and $f : M \to N$ a $C^\infty$ map. Suppose that $M$ is compact, $N$ is connected, $f$ is injective, and $df_x$ is an isomorphism for each $x \in M$. Show that $f$ is a diffeomorphism.

4. Let $u = x^2 - y^3$, $v = 3xy + y^2 - x^2$.
   (a) For which $(a, b)$ in $\mathbb{R}^2$ is there a neighborhood $U$ of $(a, b)$ such that $(U, (u, v))$ is a coordinate system?
   (b) For which real numbers $c$ is the locus $y^2 - x(x - 1)(x - c) = 0$ a submanifold of $\mathbb{R}^2$?

5. Let $T^2 = S^1 \times S^1$ be the torus, with coordinates $(x, y)$. Let $H(x, y) \in C^\infty(T^2)$ be a smooth function. Consider the flow $\phi_t$ generated by the following linear system:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial H}{\partial y} \\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x}
\end{align*}
\]

Prove that:
   (a) $\phi_t$ exists for all $t \in \mathbb{R}$.
   (b) $\phi_t^*(dx \wedge dy) = dx \wedge dy$, for all $t \in \mathbb{R}$.

6. Let $H$ be the Heisenberg group

\[
H = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid x_{12}, x_{13}, x_{23} \in \mathbb{R} \right\}
\]

of upper-diagonal $3 \times 3$ real matrices with 1’s on the diagonal. This group has natural coordinates $(x_{12}, x_{13}, x_{23})$, and it acts on itself by left translations. Let $v_{12}, v_{13}, v_{23}$ be the left-invariant vector-fields on $H$, with values at the identity $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. Consider the 2-dimensional distributions $E$ and $F$ on $H$ generated by $v_{12}, v_{13}$ and $v_{12}, v_{23}$, respectively. Show that $E$ is integrable and $F$ is not.