VANISHING RESONANCE AND REPRESENTATIONS OF LIE ALGEBRAS

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Abstract. We explore a relationship between the classical representation theory of a complex, semisimple Lie algebra $\mathfrak{g}$ and the resonance varieties $\mathcal{R}(V,K) \subset V^*$ attached to irreducible $\mathfrak{g}$-modules V and submodules $K \subset V \wedge V$. In the process, we give a precise roots-and-weights criterion insuring the vanishing of these varieties, or, equivalently, the finiteness of certain modules $\mathcal{W}(V,K)$ over the symmetric algebra on V. In the case when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, our approach sheds new light on the modules studied by Weyman and Eisenbud in the context of Green's conjecture on free resolutions of canonical curves. In the case when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{sp}_{2g}(\mathbb{C})$, our approach yields a unified proof of two vanishing results for the resonance varieties of the (outer) Torelli groups of surface groups, results which arose in recent work by Dimca, Hain, and the authors on homological finiteness in the Johnson filtration of mapping class groups and automorphism groups of free groups.

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1. Introduction

1.1. **Koszul modules and resonance varieties.** We start with a simple, yet very general construction. Let V be a non-trivial, finite-dimensional complex vector space, and let $K \subset V \wedge V$ be a subspace. To these data, we associate two objects:

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- The *Koszul module*, $\mathcal{W}(V,K)$, is a graded module over the symmetric algebra $S = \mathrm{Sym}(V)$, given by an explicit presentation involving the third Koszul differential and the inclusion map of K into $V \wedge V$.
- The *resonance variety*, $\mathcal{R}(V,K)$, is a homogeneous subvariety inside the dual vector space V^*, consisting of all elements $a \in V^*$ for which there is an element $b \in V^*$, not proportional to a, such that $a \wedge b$ belongs to the orthogonal complement $K^\perp \subseteq V^* \wedge V^*$.

These two objects are closely related: at least away from the origin, the resonance variety is the support of the Koszul module.

We investigate here conditions insuring that the resonance variety vanishes, i.e., that the Koszul module is finite-dimensional over $\mathbb{C}$. It is easily seen that $\mathcal{R}(V,K) = \{0\}$ if and only if the plane $\mathbb{P}(K^\perp)$ misses the image under the Plücker embedding in $\mathbb{P}(V^* \wedge V^*)$ of the Grassmannian of 2-planes in V^*. Thinking of K as a point in the Grassmannian of m-planes in $V \wedge V$, where $m = \dim K$, we prove in Propositions 2.6, 2.7, and 2.10 the following result.

Theorem 1.1. *For any integer m with $0 \leq m \leq \binom{n}{2}$, where $n = \dim V$, the set*

$$U_{n,m} = \big\{K \in \mathrm{Gr}_m(V \wedge V) \mid \mathcal{R}(V,K) = \{0\}\big\}$$

is Zariski open. Moreover, this set is non-empty if and only if $m \geq 2n-3$, in which case there is an integer $q = q(n,m)$ such that $\mathcal{W}_q(V,K) = 0$, for every $K \in U_{n,m}$.

1.2. **Roots, weights, and vanishing resonance.** After these general considerations, we narrow our focus, and analyze in detail the case when the vector spaces V and K are representations of a complex, semisimple Lie algebra $\mathfrak{g}$. To start with, we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of simple roots $\Delta \subset \mathfrak{h}^*$, and we denote by $(\ ,\ )$ the inner product on $\mathfrak{h}^*$ defined by the Killing form.

Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module corresponding to a dominant weight λ. In Lemmas 3.3 and 3.4, we analyze the irreducible summands in $V(\lambda) \wedge V(\lambda)$ associated to weights of the form $2\lambda - \beta$, for some $\beta \in \Delta$: such summands occur if and only if $(\lambda, \beta) \neq 0$.

Returning to the main theme, we give in Proposition 4.1 and Theorem 4.6 two very precise criteria that relate the vanishing of the resonance varieties associated to $\mathfrak{g}$-modules as above to the roots and weights of the Lie algebra $\mathfrak{g}$. To summarize these criteria, recall that each simple root $\beta \in \Delta$ gives rise to elements $x_\beta, y_\beta \in \mathfrak{g}$ and $h_\beta \in \mathfrak{h}$ which generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 1.2. *Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module, and let $K \subset V \wedge V$ be a submodule. Let $V^* = V(\lambda^*)$ be the dual module, and let v_{λ^*} be a maximal vector for V^*.*

1. *Suppose there is a root $\beta \in \Delta$ such that $(\lambda^*, \beta) \neq 0$, and suppose the vector $v_{\lambda^*} \wedge y_\beta v_{\lambda^*}$ (of weight $2\lambda^* - \beta$) belongs to $K^\perp$. Then $\mathcal{R}(V,K) \neq \{0\}$.*
2. *Suppose that $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root β. Then $\mathcal{R}(V,K) = \{0\}$.*

As a special case of this theorem, we single out in Corollary 4.7 the following situation.

Corollary 1.3. *In the setup from above, $\mathcal{R}(V,K) = \{0\}$ if and only if $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root β such that $(\lambda^*, \beta) \neq 0$.*

In the case when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, all irreducible representations are of the form $V_n = V(n\lambda_1)$, for some $n \geq 0$, where V_1 is the defining representation. Moreover, $V_n \wedge V_n$

decomposes into multiplicity-free irreps of the form $V_{2n-2-4j}$, with $j \geq 0$. The modules $W(n) = \mathcal{W}(V_n, V_{2n-2})$ have been studied by J. Weyman and D. Eisenbud, in an attempt to establish an improved version of a conjecture of M. Green regarding the syzygies of canonically embedded curves, see [5].

In Corollary 5.3, we recover and strengthen a result from [5], as follows: For any $\mathfrak{sl}_2(\mathbb{C})$-submodule $K \subset V_n \wedge V_n$, the corresponding Koszul module $\mathcal{W}(V_n, K)$ is finite-dimensional over $\mathbb{C}$ if and only if $\mathcal{W}(V_n, K)$ is a quotient of $W(n)$. The determination of the Hilbert series of these graded modules remains an interesting open problem.

1.3. **Alexander invariants, resonance, and Torelli groups.** We switch now to invariants associated to a finitely generated group G. Set $V = H_1(G, \mathbb{C})$, identify $V^* = H^1(G, \mathbb{C})$, and declare $K^\perp$ to be the kernel of the cup-product map, $\cup_G \colon V^* \wedge V^* \to H^2(G, \mathbb{C})$. In this case, the variety $\mathcal{R}(G) = \mathcal{R}(V, K)$ coincides with the first resonance variety of the group. Moreover, the Koszul module, $\mathfrak{B}(G) = \mathcal{W}(V, K)$, can be thought of as an infinitesimal version of the classical Alexander invariant, $B(G) = H_1(G', \mathbb{C})$, viewed as a module over the group algebra $\mathbb{C}[G_{\mathrm{ab}}]$, or its reduced version, $\widetilde{B}(G)$, viewed as a module over $\mathbb{C}[G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}})]$. The I-adic associated graded objects, $\operatorname{gr}_\bullet B(G)$ and $\operatorname{gr}_\bullet \widetilde{B}(G)$, are both graded modules over the polynomial algebra $\operatorname{Sym}(V)$.

Let $\operatorname{Aut}(G)$ be the automorphism group of G, and let $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ be its outer automorphism group. The Torelli group, $\mathcal{T}_G$, is the kernel of the natural homomorphism $\operatorname{Out}(G) \to \operatorname{Aut}(G_{\mathrm{ab}})$ which sends an (outer) automorphism to the induced map on the abelianization. Using our machinery, we give a unified proof of two key results from [3] and [16] concerning vanishing resonance for the Torelli groups of surface groups, and recover some additional information on their Alexander invariants.

First, we consider the free group on $n \geq 4$ generators; its Torelli group, $\mathrm{OA}_n = \mathcal{T}_{F_n}$, is finitely generated. Work of Andreadakis, Cohen–Pakianathan, Farb, and Kawazumi shows that the natural action of $\mathrm{GL}_n(\mathbb{Z})$ on $H_1(\mathrm{OA}_n, \mathbb{Z})$ defines an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module structure on $V = H_1(\mathrm{OA}_n, \mathbb{C})$. Furthermore, work of Pettet [17] identifies V^* with $V(\lambda_1 + \lambda_{n-2})$ and $K^\perp = \ker(\cup_{\mathrm{OA}_n})$ with $V(\lambda_1 + \lambda_{n-2} + \lambda_{n-1})$, where $\lambda_1, \dots, \lambda_{n-1}$ are the fundamental dominant weights for $\mathfrak{sl}_n(\mathbb{C})$. With this notation, we prove in Theorem 7.1 and Corollary 7.3 the following result.

Theorem 1.4. *For each $n \geq 4$, the resonance variety $\mathcal{R}(\mathrm{OA}_n) = \mathcal{R}(V, K)$ vanishes. Moreover, $\dim \mathcal{W}(V, K) < \infty$ and $\dim \operatorname{gr}_q B(\mathrm{OA}_n) \leq \dim \mathcal{W}_q(V, K)$, for all $q \geq 0$.*

Next, we consider the fundamental group of a Riemann surface of genus $g \geq 3$. The corresponding Torelli group, $\mathcal{T}_g = \mathcal{T}_{\pi_1(\Sigma_g)}$, is known to be finitely generated. Work of D. Johnson [12] shows that the natural $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on $H_1(\mathcal{T}_g, \mathbb{Z})$ defines an irreducible $\mathfrak{sp}_{2g}(\mathbb{C})$-module structure on $V = H_1(\mathcal{T}_g, \mathbb{C})$. Work of Hain [8] identifies V^* with $V(\lambda_3)$ and $K^\perp$ with $V(2\lambda_2) \oplus V(0)$, where $\lambda_1, \dots, \lambda_g$ are the fundamental dominant weights for $\mathfrak{sp}_{2g}(\mathbb{C})$. In Theorem 7.4 and Corollary 7.6, we prove the following result.

Theorem 1.5. *For each $g \geq 4$, the resonance variety $\mathcal{R}(\mathcal{T}_g) = \mathcal{R}(V, K)$ vanishes. Moreover, if $g \geq 6$, then $\dim \widetilde{B}(\mathcal{T}_g) = \dim \mathcal{W}(V, K) < \infty$.*

2. Koszul modules and resonance varieties

2.1. **Koszul modules.** Let V be a non-zero, finite-dimensional vector space over $\mathbb{C}$. Let $S = \operatorname{Sym}(V)$ be the symmetric algebra on V, let $\bigwedge V$ be the exterior algebra on V, and let

$(S\otimes_{\mathbb{C}}\bigwedge V,\delta)$ be the Koszul resolution, where the differential $\delta_p\colon S\otimes_{\mathbb{C}}\bigwedge^p V\to S\otimes_{\mathbb{C}}\bigwedge^{p-1}V$ is the S-linear map given on basis elements by

$$\delta_p(v_{i_1}\wedge\cdots\wedge v_{i_p})=\sum_{j=1}^{p}(-1)^{j-1}v_{i_j}\otimes(v_{i_1}\wedge\cdots\wedge\widehat{v_{i_j}}\wedge\cdots\wedge v_{i_p}). \tag{1}$$

Definition 2.1. The *Koszul module* associated to a linear subspace $K\subseteq V\wedge V$ is the S-module $\mathcal{W}(V,K)$ presented as

$$S\otimes_{\mathbb{C}}\big(\textstyle\bigwedge^3 V\oplus K\big)\xrightarrow{\delta_3+\mathrm{id}\otimes\iota}S\otimes_{\mathbb{C}}\bigwedge^2 V\twoheadrightarrow\mathcal{W}(V,K)\,, \tag{2}$$

where the group $\bigwedge^3 V$ is in degree 1, the groups K and $\bigwedge^2 V$ are in degree 0, and $\iota\colon K\to V\wedge V$ is the inclusion map.

Alternatively, we may write

$$\mathcal{W}(V,K)=\operatorname{im}(\delta_2)/\operatorname{im}(\delta_2\circ(\mathrm{id}\otimes\iota)). \tag{3}$$

The S-module $\mathcal{W}=\mathcal{W}(V,K)$ inherits a natural grading from the free S-module $S\otimes_{\mathbb{C}}\bigwedge^2 V$; we will denote by $\mathcal{W}_q$ its q-th graded piece.

Remark 2.2. Note that $\mathcal{W}$ is generated as an S-module by its degree 0 piece, $\mathcal{W}_0=(V\wedge V)/K$. Thus, the S-module $\mathcal{W}(V,K)$ is a finite-dimensional $\mathbb{C}$-vector space if and only if there is an integer $q\geq 0$ such that $\mathcal{W}_q(V,K)=0$.

The above construction enjoys some nice functoriality properties. For instance, suppose $(V'\wedge V',K')$ is another pair of vector spaces as above, and suppose there is a linear map $\varphi\colon V\to V'$ such that $\varphi\wedge\varphi\colon V\wedge V\to V'\wedge V'$ takes K to K'. Write $S'=\operatorname{Sym}(V')$, and let $\operatorname{Sym}(\varphi)\colon S\to S'$ be the induced ring morphism. The commuting diagram

$$\begin{array}{ccccc}
S\otimes_{\mathbb{C}}\big(\bigwedge^3 V\oplus K\big) & \xrightarrow{\delta_3+\mathrm{id}\otimes\iota} & S\otimes_{\mathbb{C}}\bigwedge^2 V & \twoheadrightarrow & \mathcal{W}(V,K)\\
\downarrow{\scriptstyle \operatorname{Sym}(\varphi)\otimes(\bigwedge^3\varphi\oplus\bigwedge^2\varphi|_K)} & & \downarrow{\scriptstyle \operatorname{Sym}(\varphi)\otimes\bigwedge^2\varphi} & & \downarrow{\scriptstyle \mathcal{W}(\varphi)}\\
S'\otimes_{\mathbb{C}}\big(\bigwedge^3 V'\oplus K'\big) & \xrightarrow{\delta_3'+\mathrm{id}\otimes\iota'} & S'\otimes_{\mathbb{C}}\bigwedge^2 V' & \twoheadrightarrow & \mathcal{W}(V',K').
\end{array} \tag{4}$$

defines a $\operatorname{Sym}(\varphi)$-equivariant map $\mathcal{W}(\varphi)$ between the respective Koszul modules. Note that, if φ is surjective, then $\mathcal{W}(\varphi)$ is also surjective.

2.2. **Resonance varieties.** Let $V^*=\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ be the dual vector space, and let $K^\perp\subseteq V^*\wedge V^*=(V\wedge V)^*$ be the linear subspace consisting of all functionals that vanish identically on the subspace $K\subseteq V\wedge V$.

Definition 2.3. The *resonance variety* associated to the pair $(V\wedge V,K)$ is defined as

$$\mathcal{R}(V,K)=\{a\in V^*\mid\exists\, b\in V^*\text{ such that }a\wedge b\neq 0\text{ and }a\wedge b\in K^\perp\}\cup\{0\}. \tag{5}$$

It is readily seen that $\mathcal{R}(V,K)$ is a conical, Zariski-closed subset of the affine space V^*. For instance, if $K=0$ and if $\dim V>1$, then $\mathcal{R}(V,0)=V^*$. At the other extreme, $\mathcal{R}(V,V\wedge V)=\{0\}$.

This construction also enjoys some pleasant functoriality properties. For instance, suppose $\varphi\colon V\twoheadrightarrow V'$ is a surjective linear map inducing a morphism from $(V\wedge V,K)$ to $(V'\wedge V',K')$. Then, the linear map $\varphi^*\colon(V')^*\hookrightarrow V^*$ restricts to an embedding $\mathcal{R}(\varphi)\colon\mathcal{R}(V',K')\hookrightarrow\mathcal{R}(V,K)$ between the respective resonance varieties.

After identifying the ring $S = \operatorname{Sym}(V)$ with the coordinate ring of V^*, we may view the support of the S-module $\mathcal{W}(V,K)$ as a subvariety of the affine space V^*. The next lemma connects this support variety to the resonance variety of the pair $(V \wedge V, K)$.

Lemma 2.4. *Let $K \subseteq V \wedge V$ be a linear subspace. Then:*

(1) *Away from the origin, $\mathcal{R}(V,K) = \operatorname{supp}(\mathcal{W}(V,K))$.*
(2) *$\dim_{\mathbb{C}} \mathcal{W}(V,K) < \infty$ if and only if $\mathcal{R}(V,K) = \{0\}$.*

Proof. Let $\delta_p(a)\colon \bigwedge^p V \to \bigwedge^{p-1} V$ be the evaluation of the p-th Koszul differential at a non-zero element $a \in V^*$. Using formula (3) and the usual description of support in terms of Fitting ideals, we deduce that

$$a \in \operatorname{supp}(\mathcal{W}(V,K)) \iff \dim_{\mathbb{C}}(\operatorname{im}(\delta_2(a))/\operatorname{im}(\delta_2(a)\circ\iota)) \geq 1. \tag{6}$$

Let $\delta_p^*(a)$ be the transpose of the matrix $\delta_p(a)$. Direct calculation shows that $\delta_2^*(a)\colon V^* \to V^* \wedge V^*$ is the map $b \mapsto a \wedge b$. Hence,

$$a \in \mathcal{R}(V,K) \iff \dim_{\mathbb{C}}(\ker(\pi\circ\delta_2^*(a))/\ker(\delta_2^*(a))) \geq 1, \tag{7}$$

where $\pi\colon V^* \wedge V^* \to (V^* \wedge V^*)/K^\perp$ is the canonical projection.

Upon identifying the matrix of π with the transpose of the matrix of ι, part (1) follows by comparing formulas (6) and (7). Finally, part (2) follows from part (1) by standard commutative algebra. □

2.3. **Grassmannians and resonance.** As usual, let $K \subseteq V \wedge V$ be a linear subspace. Setting $m = \dim K$, we may view K as a point in the Grassmannian $\operatorname{Gr}_m(V \wedge V)$ of m-planes in $V \wedge V$, endowed with the Zariski topology.

Now let $K^\perp \subseteq V^* \wedge V^*$ be the orthogonal complement. Then $\mathbb{P}(K^\perp)$ may be viewed as a codimension m projective subspace in $\mathbb{P}(V^* \wedge V^*)$.

Lemma 2.5. *Let $\operatorname{Gr}_2(V^*) \hookrightarrow \mathbb{P}(V^* \wedge V^*)$ be the Plücker embedding. Then,*

$$\mathcal{R}(V,K) = \{0\} \iff \mathbb{P}(K^\perp) \cap \operatorname{Gr}_2(V^*) = \emptyset.$$

Proof. Follows straight from the definition of resonance. □

Now denote the dimension of V by n, and fix an isomorphism $V = \mathbb{C}^n$. Consider the following subset of the Grassmannian of m-planes in $V \wedge V = \mathbb{C}^{\binom{n}{2}}$:

$$U_{n,m} = \big\{K \in \operatorname{Gr}_m(V \wedge V) \mid \mathcal{R}(V,K) = \{0\}\big\}. \tag{8}$$

Proposition 2.6. *For any integer m with $0 \leq m \leq \binom{n}{2}$, where $n = \dim V$, the set $U_{n,m}$ is a Zariski open subset of $\operatorname{Gr}_m(V \wedge V)$.*

Proof. Let $\operatorname{Gr}^m(V^* \wedge V^*)$ be the Grassmannian of codimension m planes in $V^* \wedge V^*$. In view of Lemma 2.5, under the isomorphism $\operatorname{Gr}_m(V \wedge V) \xrightarrow{\simeq} \operatorname{Gr}^m(V^* \wedge V^*)$, $K \leftrightarrow K^\perp$, the set $U_{n,m}$ corresponds to the set

$$\mathcal{U}_n^m = \big\{K^\perp \in \operatorname{Gr}^m(V^* \wedge V^*) \mid \mathbb{P}(K^\perp) \cap \operatorname{Gr}_2(V^*) = \emptyset\big\}. \tag{9}$$

Clearly, $\mathcal{U}_n^m$ is the complement in $\operatorname{Gr}^m(V^* \wedge V^*)$ to the set of codimension m planes incident to $\operatorname{Gr}_2(V^*)$. As is well-known (see e.g. [9]), this latter set is Zariski closed. The desired conclusion follows at once. □

The next result identifies the threshold value of m (as a function of n) for which the set $U_{n,m}$ is non-empty.

Proposition 2.7. *With notation as above,*

$$U_{n,m} \neq \emptyset \iff m \geq 2n-3.$$

Proof. The Grassmannian variety $\mathbb{G} = \mathrm{Gr}_2(\mathbb{C}^n)$ may be viewed as a smooth, irreducible subvariety of the projective space $\mathbb{P} = \mathbb{P}\big(\mathbb{C}^{\binom{n}{2}}\big)$. Thus, if $W = V(f)$ is a hypersurface in $\mathbb{P}$, then $\dim(\mathbb{G} \cap W) \geq \dim(\mathbb{G}) - 1$, with equality achieved if $f \notin I(\mathbb{G})$.

Let $\mathbb{K} = \mathbb{P}(K^\perp)$ be a codimension m projective subspace in $\mathbb{P}$. Then

$$\dim(\mathbb{G} \cap \mathbb{K}) \geq \dim \mathbb{G} - \operatorname{codim} \mathbb{K} = 2(n-2) - m.$$

Thus, if $m < 2n-3$, then $\dim(\mathbb{G} \cap \mathbb{K}) \geq 0$, and so $\mathbb{G} \cap \mathbb{K} \neq \emptyset$, showing that $K \notin U_{n,m}$.

Now suppose $m \geq 2n-3$. Then, there is a codimension m projective subspace $\mathbb{K} \subset \mathbb{P}$ which intersects $\mathbb{G}$ in the empty set. Thus, $K \in U_{n,m}$. □

2.4. **The graded pieces of a Koszul module.** As before, let $K \subseteq V \wedge V$ be a subspace, and let $K^\perp \subseteq V^* \wedge V^*$ be its orthogonal complement. Set

$$A(V,K) = \bigwedge V^* / \mathrm{ideal}(K^\perp). \tag{10}$$

Clearly, $A = A(V,K)$ is a finite-dimensional, graded-commutative $\mathbb{C}$-algebra; moreover, A is quadratic (i.e., it has a presentation with generators in degree 1 and relations in degree 2), and satisfies $a^2 = 0$, for all $a \in A^1$.

Using a result of Fröberg and Löfwall [6, Theorem 4.1], as recast in [15, Proposition 2.3], as well as [14, Theorem 6.2], we may reinterpret the (duals of the) graded pieces of the Koszul module $\mathcal{W}(V,K)$ in terms of the linear strand in an appropriate Tor module.

Proposition 2.8. *For all $q \geq 0$,*

$$\mathcal{W}_q(V,K)^* \cong \mathrm{Tor}^{\bigwedge V^*}_{q+1}(A(V,K), \mathbb{C})_{q+2}.$$

Example 2.9. Suppose the subspace $K \subseteq V \wedge V$ admits a monomial basis, that is to say, suppose there is a basis $\mathsf{V} = \{v_1, \dots, v_n\}$ for V and a subset $\mathsf{E} \subseteq \binom{\mathsf{V}}{2}$ such that the set $\{v_i \wedge v_j \mid \{i,j\} \in \mathsf{E}\}$ is a basis for K. Then the algebra $A(V,K)$ is the exterior Stanley-Reisner ring associated to the graph Γ with vertex set V and edge set E. Using Proposition 2.8, we showed in [15, Theorem 4.1] that the Hilbert series of the graded module $\mathcal{W}(V,K)$ is given by

$$\sum_{q=0}^{\infty} \dim \mathcal{W}_q(V,K) t^{q+2} = Q_\Gamma\left(\frac{t}{1-t}\right). \tag{11}$$

Here, $Q_\Gamma(t) = \sum_{j \geq 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with coefficient $c_j(\Gamma)$ equal to $\sum_{\mathsf{W} \subset \mathsf{V}:\ |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}})$, where $\tilde{b}_0(\Gamma_{\mathsf{W}})$ is one less than the number of components of the full subgraph on W.

2.5. **A vanishing range.** Now suppose the subspace K belongs to the Zariski open set $U_{n,m} \subseteq \mathrm{Gr}_m(V \wedge V)$ from (8). Then, as we shall show in the next proposition, all the graded pieces of the Koszul module $\mathcal{W}(V,K)$ vanish beyond a range which depends only on the integers $n = \dim V$ and $m = \dim K$.

Proposition 2.10. *For each $m \geq 2n-3$, there is an integer $q = q(n,m)$ such that $\mathcal{W}_q(V,K) = 0$, for every $K \in U_{n,m}$.*

Proof. For each $q \geq 0$, consider the following subset of the Grassmannian of m-planes in $V \wedge V$:

$$U^q_{n,m} = \big\{K \in \mathrm{Gr}_m(V \wedge V) \mid \mathcal{W}_q(V, K) = 0\big\}. \tag{12}$$

By definition, the vector space $\mathcal{W}_q(V, K)$ is the cokernel of a matrix whose entries depend continuously on the m-plane $K \subset V \wedge V$. Thus, the set $U^q_{n,m}$ where those matrices have maximal rank is a Zariski open subset of $\mathrm{Gr}_m(V \wedge V)$.

Now, Lemma 2.4(2) and definitions (8) and (12) together imply that

$$U_{n,m} = \bigcup_{q \geq 0} U^q_{n,m}. \tag{13}$$

Since the graded $\mathrm{Sym}(V)$-module $\mathcal{W}(V, K)$ is generated in degree zero, the union in (13) is an increasing union.

Putting things together, we conclude that $U_{n,m}$ is an increasing union of Zariski open sets. Thus, there must exist an integer q (depending on n and m) such that $U^q_{n,m} = U_{n,m}$. This finishes the proof. □

For instance, if $\binom{n}{2} - m \leq 1$, then $q(n, m) = \binom{n}{2} - m$. In general, though, the determination of the integer $q(n, m)$ seems to be a challenging, yet interesting problem. We will come back to this point in Section 5.

3. Some representation theory

In this section, we analyze the decomposition into irreducible summands of the second exterior power of an irreducible representation of a complex, semisimple Lie algebra.

3.1. **Semisimple Lie algebras.** We start by reviewing some basic notions from [10]. Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra over $\mathbb{C}$. A choice of Cartan subalgebra, $\mathfrak{h} \subset \mathfrak{g}$, yields a root system for $\mathfrak{g}$: by definition, this is the set $\Phi \subset \mathfrak{h}^*$ of eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

The Killing form defines an inner product, $(\,,\,)$ on $\mathfrak{h}$. It also sets up an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$, which gives by transport of structure an inner product on $\mathfrak{h}^*$, denoted also by $(\,,\,)$. Given $\alpha, \beta \in \mathfrak{h}^*$, write

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)}. \tag{14}$$

Inside the root system Φ, fix a set of positive roots Φ^+, so that $\Phi = \Phi^+ \sqcup \Phi^-$. This choice determines a direct sum decomposition, $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$, where $\mathfrak{g}^\pm = \bigcup_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$, and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$.

Let $\Delta \subset \Phi^+$ be the corresponding set of simple roots. Note that $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, where $\mathfrak{h}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space spanned by Δ. Define the height function, $\mathrm{ht}\colon \mathfrak{h}^*_{\mathbb{Q}} \to \mathbb{Q}$, as the $\mathbb{Q}$-linear function given by $\mathrm{ht}(\alpha) = 1$ for every $\alpha \in \Delta$.

Inside the rational vector space $\mathfrak{h}^*_{\mathbb{Q}}$, there are two subsets worth singling out: the integral weight lattice, Λ, defined as

$$\Lambda = \big\{\lambda \in \mathfrak{h}^*_{\mathbb{Q}} \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}, \text{ for all } \alpha \in \Delta\big\}, \tag{15}$$

and the positive cone, C^+, defined as

$$C^+ = \Big\{\gamma \in \mathfrak{h}^*_{\mathbb{Q}} \mid \ \gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha, \text{ with } c_\alpha \in \mathbb{Z}_{\geq 0}\Big\}. \tag{16}$$

Clearly, $\Phi^+ \subset C^+$. The positive cone defines a partial order on the weight lattice Λ: we say $\nu \preceq \tau$ if $\tau - \nu \in C^+$.

3.2. **Irreducible representations.** Let $\Lambda^+ \subset \Lambda$ be the set of dominant weights, defined by

$$\lambda \in \Lambda^+ \iff \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0},\ \forall \alpha \in \Delta. \tag{17}$$

This set parametrizes all finite-dimensional, irreducible representations of the Lie algebra $\mathfrak{g}$: for each dominant weight $\lambda \in \Lambda^+$, the corresponding representation is denoted by $V(\lambda)$. A non-zero vector $v \in V(\lambda)$ is said to be a maximal vector (of weight λ) if $\mathfrak{g}^+ \cdot v = 0$. Such a vector is uniquely determined (up to non-zero scalars), and will be denoted by v_λ.

The dual representation, $V(\lambda)^*$, can be written as $V(\lambda)^* = V(\lambda^*)$, for a unique weight $\lambda^* \in \Lambda^+$. We will denote a maximal vector for $V(\lambda^*)$ by v_{λ^*}.

Given a finite-dimensional $\mathfrak{g}$-module U, we associate to it two sets of weights, as follows.

Definition 3.1. The set of dominant weights occurring in U is

$$\mathrm{Weight}^+(U) = \{\mu \in \Lambda^+ \mid U \text{ contains a summand isomorphic to } V(\mu)\}$$

Denote by U_μ the eigenspace of the $\mathfrak{h}$-action on U, with eigenvalue μ.

Definition 3.2. The set of weights occurring in U is

$$\mathrm{Weight}(U) = \{\mu \in \Lambda \mid U_\mu \neq 0\}.$$

It follows from the definitions that $\mathrm{Weight}^+(U) \subseteq \mathrm{Weight}(U)$. When viewed as an $\mathfrak{h}$-module by restriction, U decomposes as

$$U = \bigoplus_{\mu \in \mathrm{Weight}(U)} U_\mu. \tag{18}$$

3.3. **A weight test.** Let $V(\lambda)$ be an irreducible $\mathfrak{g}$-module corresponding to a dominant weight λ. The next lemma provides a simple criterion for the existence of irreducible summands with certain prescribed weights in the second exterior power of $V(\lambda)$.

Lemma 3.3. *The representation $V(\lambda) \wedge V(\lambda)$ contains a direct summand isomorphic to $V(2\lambda - \beta)$, for some simple root β, if and only if $(\lambda, \beta) \neq 0$. When it exists, such a summand is unique.*

Proof. First suppose $2\lambda - \beta$ is a dominant weight, for some $\beta \in \Delta$. Then $(2\lambda - \beta, \alpha) \geq 0$, for every $\alpha \in \Delta$. In particular, $(2\lambda - \beta, \beta) \geq 0$, which forces $(\lambda, \beta) \neq 0$, by the non-degeneracy of the Killing form.

Conversely, suppose that $(\lambda, \beta) \neq 0$. Then, by [1, Ch. VIII, §7, Exercice 17], the second exterior power of $V(\lambda)$ contains a unique direct summand of type $V(2\lambda - \beta)$. □

3.4. **A decomposable maximal vector.** For every $\alpha \in \Phi^+$, there exist elements $x_\alpha, y_\alpha \in \mathfrak{g}$ and $h_\alpha \in \mathfrak{h}$ such that $\{x_\alpha, y_\alpha, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C})$. We refer to [10, Lemma 21.2] for basic commutation relations among those elements. In particular, $[x_\alpha, y_\beta] = 0$, for all distinct $\alpha, \beta \in \Delta$. It is readily seen that $\mathfrak{g}^+$ is the subspace generated by $\{x_\alpha : \alpha \in \Phi^+\}$.

Let $\lambda \in \Lambda^+$ be a dominant weight, and let v_λ be a maximal vector for $V(\lambda)$. Given a simple root $\beta \in \Delta$, let $K(\lambda, \beta)$ be the $\mathfrak{g}$-invariant subspace of $V(\lambda) \wedge V(\lambda)$ spanned by the vector $v_\lambda \wedge y_\beta v_\lambda$.

Lemma 3.4. *Let $\beta \in \Delta$ be a simple root, and let $V = V(\lambda)$ be an irreducible representation, with maximal vector v_λ. If $(\lambda, \beta) \neq 0$, then $v_\lambda \wedge y_\beta v_\lambda$ is a maximal vector for $K(\lambda, \beta)$. In particular, $K(\lambda, \beta)$ is an irreducible $\mathfrak{g}$-module, of highest weight $2\lambda - \beta \in \mathrm{Weight}^+(V \wedge V)$.*

Proof. Set $\mu := 2\lambda - \beta$, and consider the vector $y_\beta v_\lambda \in V(\lambda)$. Since $x_\beta v_\lambda = 0$, we have that

$$x_\beta y_\beta v_\lambda = h_\beta v_\lambda = \lambda(h_\beta) v_\lambda = 2\frac{(\lambda, \beta)}{(\beta, \beta)} v_\lambda \neq 0, \tag{19}$$

and thus $y_\beta v_\lambda \neq 0$. Moreover, the vector $y_\beta v_\lambda$ has weight $\lambda - \beta \neq \lambda$. Thus, v_λ and $y_\beta v_\lambda$ are linearly independent in $V(\lambda)$.

Now set $k_\mu = v_\lambda \wedge y_\beta v_\lambda$. By the above, $k_\mu \neq 0$. Note that k_μ has weight $\lambda + (\lambda - \beta) = \mu$. To check the remaining condition for maximality, that is, $\mathfrak{g}^+ k_\mu = 0$, it is enough to show that $x_\alpha k_\mu = 0$ for all $\alpha \in \Delta$. If $\beta \neq \alpha$, then $[x_\alpha, y_\beta] = 0$, and so $x_\alpha k_\mu = v_\lambda \wedge y_\beta x_\alpha v_\lambda = 0$. Moreover, $x_\beta k_\mu = v_\lambda \wedge x_\beta y_\beta v_\lambda = 0$, by (19).

Thus, k_μ is a maximal vector (of weight μ) for $K(\lambda, \beta)$. In other words, $K(\lambda, \beta) \cong V(\mu)$, and we conclude that $\mu \in \mathrm{Weight}^+(V \wedge V)$. □

4. Weights and resonance

In this section, we prove our main results concerning the resonance varieties associated to suitable representations of complex semisimple Lie algebras.

4.1. **A non-vanishing criterion.**

We start with a test insuring that the resonance variety associated to a $\mathfrak{g}$-invariant subspace $K \subset V \wedge V$ does not vanish.

Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module, corresponding to a dominant weight $\lambda \in \Lambda^+$, and let v_λ be a maximal vector in V. Given a simple root $\beta \in \Delta$, recall that $K(\lambda, \beta)$ denotes the $\mathfrak{g}$-invariant subspace of $V \wedge V$ spanned by the vector $v_\lambda \wedge y_\beta v_\lambda$.

Proposition 4.1. *Let $K \subset V \wedge V$ be a $\mathfrak{g}$-invariant subspace. Suppose $K^\perp \supseteq K(\lambda^*, \beta)$, for some $\beta \in \Delta$ with $(\lambda^*, \beta) \neq 0$. Then v_{λ^*} belongs to $\mathcal{R}(V, K)$.*

Proof. Since $(\lambda^*, \beta) \neq 0$, Lemma 3.4 guarantees that the $\mathfrak{g}$-module $K(\lambda^*, \beta)$ is irreducible, and has maximal vector $v_{\lambda^*} \wedge y_\beta v_{\lambda^*}$, of weight $2\lambda^* - \beta$. Thus, $v_{\lambda^*} \wedge y_\beta v_{\lambda^*}$ is a non-zero element in $K^\perp$. Therefore, $v_{\lambda^*} \in \mathcal{R}(V(\lambda), K)$. □

4.2. **Maximal vectors in resonance varieties.**

We record now a basic symmetry property enjoyed by resonance varieties.

Lemma 4.2. *Let G be a group, and let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. Let $K \subset V \wedge V$ be a G-invariant subspace. Then, the corresponding resonance variety, $\mathcal{R}(V, K)$, is a G-invariant subset of V^*.*

Proof. Let $a \in V^*$ be a non-zero element of $\mathcal{R}(V, K)$. By definition, this means there is an element $b \in V^*$ such that $a \wedge b \neq 0$ and $a \wedge b \in K^\perp$.

Now let g be an element of G. Since G acts diagonally on $V^* \wedge V^*$, and since $K^\perp$ is a G-invariant subspace of $V^* \wedge V^*$, we have that $ga \wedge gb \neq 0$ and $ga \wedge gb \in K^\perp$. This shows that $ga \in \mathcal{R}(V, K)$, and we are done. □

Next, we recall a lemma from [3].

Lemma 4.3 ([3]). *Let G be a complex, semisimple, linear algebraic group, with Lie algebra $\mathfrak{g}$. Let U be an irreducible, rational representation of G. If $\mathcal{R}$ is a Zariski-closed, G-invariant cone in U, and if $\mathcal{R} \neq \{0\}$, then $\mathcal{R}$ contains a maximal vector for the $\mathfrak{g}$-module U.*

The proof of this lemma is based on the classical Borel fixed point theorem (cf. [11]), which insures that the action of the Borel subgroup B with Lie algebra $\mathfrak{h} \oplus \mathfrak{g}^+$ has a fixed point in the projectivization $\mathbb{P}(\mathcal{R}) \subseteq \mathbb{P}(U)$.

Using the above two lemmas, and some standard facts about Lie groups and Lie algebras from Serre's monograph [18], we can now pin down a maximal vector in each non-zero resonance variety associated to an irreducible $\mathfrak{g}$-representation.

Proposition 4.4. *Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, let V be an irreducible $\mathfrak{g}$-module, and let $K \subseteq V \wedge V$ be a $\mathfrak{g}$-submodule. If the resonance variety $\mathcal{R} = \mathcal{R}(V, K)$ is not equal to $\{0\}$, then $\mathcal{R}$ contains a maximal vector for the $\mathfrak{g}$-module V^*.*

Proof. As shown in [18], there is a unique simply-connected, complex, semisimple, linear algebraic group G such that $\mathrm{Lie}(G) = \mathfrak{g}$. Furthermore, the vector space V supports a rational, irreducible representation of G, such that the associated infinitesimal representation of $\mathrm{Lie}(G)$ coincides with the given $\mathfrak{g}$-module structure on V.

It follows that the subspace $K \subseteq V \wedge V$ is G-invariant. Hence, by Lemma 4.2, the set $\mathcal{R} = \mathcal{R}(V, K)$ is also G-invariant. Of course, $\mathcal{R}$ is a homogeneous variety, and thus a conical subset of V^*; furthermore, $\mathcal{R} \neq \{0\}$, by assumption. Thus, all the hypothesis of Lemma 4.3 have been verified, and we conclude that $\mathcal{R}$ contains a maximal vector for V^*. □

4.3. **Decomposing a maximal vector.** The following lemma generalizes Lemmas 3.5 from [3] and 9.6 from [16]. Although the proof follows in rough outline the proofs of those two previous results, we include here full details, for the reader's convenience.

Lemma 4.5. *Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module, and let v_{λ^*} be a maximal vector for V^*. Let K be a $\mathfrak{g}$-invariant subspace of $V \wedge V$. Suppose v_{λ^*} belongs to $\mathcal{R}(V, K)$. Then, we may find an element $w \in V^*$ such that if we set $u = v_{\lambda^*} \wedge w$, we have $u \in K^\perp$, $u \neq 0$, and $\mathfrak{g}^+ \cdot u = 0$.*

Proof. Let $\ell\colon V^* \to V^* \wedge V^*$ denote left-multiplication by the element v_{λ^*} in the exterior algebra on V^*. By definition of resonance, the vector v_{λ^*} belongs to $\mathcal{R}(V, K)$ if and only if $\mathrm{im}(\ell) \cap K^\perp \neq 0$. The Lie algebra $\mathfrak{g}^+$ annihilates the vector v_{λ^*}; hence, the linear map ℓ is $\mathfrak{g}^+$-equivariant. Thus, all we need to show is that the $\mathfrak{g}^+$-module $\mathrm{im}(\ell) \cap K^\perp$ contains a non-zero vector u annihilated by $\mathfrak{g}^+$.

This claim follows from Engel's theorem (cf. [10, Theorem 3.3]), provided each element of $\mathfrak{g}^+$ acts nilpotently on $K^\perp$. To verify this nilpotence property, we may assume $K^\perp$ is $\mathfrak{g}$-irreducible, the general case following easily from this one.

The Lie algebra $\mathfrak{g}^+$ decomposes into a direct sum of 1-dimensional vector spaces, each one spanned by an element x_α, with α running through the set of positive roots, Φ^+. Let μ be the dominant weight corresponding to $K^\perp$. Let $K^\perp_\nu$ be a non-trivial weight space for $K^\perp$, and let $\alpha_1, \dots, \alpha_r \in \Phi^+$. Then

$$x_{\alpha_1} \cdots x_{\alpha_r}(K^\perp_\nu) \subseteq K^\perp_{\nu'}, \tag{20}$$

where $\nu' = \nu + \sum_{i=1}^r \alpha_i$. Suppose $K^\perp_{\nu'} \neq 0$; then, by maximality of μ, we may write $\nu' = \mu - \beta$, for some element β in the positive cone, see [10, Theorem 20.2(b)]. Hence,

$\sum_{i=1}^{r} \alpha_i = \mu - \nu - \beta$. Since $\alpha_i \in \Phi^+$ and $\beta \in C^+$, we find that

$$r \leq \operatorname{ht}\Big(\sum_{i=1}^{r} \alpha_i\Big) \leq \operatorname{ht}(\mu - \nu). \tag{21}$$

Choosing an integer r strictly greater than $\operatorname{ht}(\mu - \nu)$, for all $\nu \in \operatorname{Weight}(K^\perp)$, formula (21) guarantees that $K^\perp_{\nu'} = 0$. From (20), it follows that each element of $\mathfrak{g}^+$ acts nilpotently on $K^\perp$. This completes the proof. □

4.4. **A vanishing criterion.** We now provide a test insuring that the resonance variety associated to a $\mathfrak{g}$-invariant subspace $K \subset V \wedge V$ does vanish.

Theorem 4.6. *Let V be an irreducible $\mathfrak{g}$-module of highest weight λ, and let K be a $\mathfrak{g}$-submodule of $V \wedge V$. Suppose that $2\lambda^* - \mu \notin \Delta$, for any $\mu \in \operatorname{Weight}^+(K^\perp)$. Then $\mathcal{R}(V, K) = \{0\}$.*

Proof. Suppose $\mathcal{R} = \mathcal{R}(V, K)$ is non-zero. Then, by Proposition 4.4, the variety $\mathcal{R} \subset V^*$ contains a maximal vector v_{λ^*} for the irreducible $\mathfrak{g}$-module $V^* = V(\lambda^*)$. We will use this fact, together with our assumption on the weight λ^*, to derive a contradiction.

Lemma 4.5 yields an element $w \in V^*$ such that, if we set $u = v_{\lambda^*} \wedge w$, then

$$u \in K^\perp, \quad u \neq 0, \quad \mathfrak{g}^+ \cdot u = 0. \tag{22}$$

Let $V^* = \bigoplus_{\nu \in \operatorname{Weight}(V^*)} V^*_\nu$ be the weight decomposition of V^* under the action of $\mathfrak{h}$; accordingly, write $w = \sum_\nu w_\nu$, where each w_ν belongs to V^*_ν, and thus has weight ν. Finally, set

$$u_{\lambda^*+\nu} = v_{\lambda^*} \wedge w_\nu. \tag{23}$$

Note that $u_{\lambda^*+\nu}$ has weight $\lambda^* + \nu$. Hence, $u = \sum_\nu u_{\lambda^*+\nu}$ is the weight decomposition of the vector $u \in K^\perp$. Since $K^\perp$ is a $\mathfrak{g}$-submodule of $V^* \wedge V^*$, it follows that each $u_{\lambda^*+\nu}$ belongs to $K^\perp$.

Now, for each $\alpha \in \Phi^+$, we have that $x_\alpha u = 0$; hence, $x_\alpha u_{\lambda^*+\nu} = 0$ for all ν, by uniqueness of the weight decomposition. Finally, since $u \neq 0$, one of the components of u must be non-zero. Therefore, there is an index ν such that

$$u_{\lambda^*+\nu} \in K^\perp, \quad u_{\lambda^*+\nu} \neq 0, \quad \mathfrak{g}^+ \cdot u_{\lambda^*+\nu} = 0. \tag{24}$$

Set $\mu := \lambda^* + \nu$. By (24), then,

$$\mu \in \operatorname{Weight}^+(K^\perp). \tag{25}$$

For any simple root $\alpha \in \Delta$, we have that

$$0 = x_\alpha u_{\lambda^*+\nu} = x_\alpha(v_{\lambda^*} \wedge w_\nu) = v_{\lambda^*} \wedge x_\alpha w_\nu, \tag{26}$$

and thus $x_\alpha w_\nu \in \mathbb{C} \cdot v_{\lambda^*}$. Suppose there is a simple root β such that $x_\beta w_\nu \neq 0$. A simple weight inspection reveals that

$$\beta + \nu = \lambda^*. \tag{27}$$

Putting together (25) and (27), we obtain that $2\lambda^* - \mu = \beta \in \Delta$, with $\mu \in \operatorname{Weight}^+(K^\perp)$, thereby contradicting our hypothesis. Therefore, $x_\alpha w_\nu = 0$, for all $\alpha \in \Delta$, i.e., $\mathfrak{g}^+ \cdot w_\nu = 0$.

On the other hand, we also know that $w_\nu \neq 0$; thus, w_ν is a maximal vector in $V^* = V(\lambda^*)$. Hence, $w_\nu \in \mathbb{C} \cdot v_{\lambda^*}$. Therefore, $u_{\lambda^*+\nu} = v_{\lambda^*} \wedge w_\nu = 0$, contradicting (24). Thus, $\mathcal{R} = \{0\}$. □

Corollary 4.7. *Let V be an irreducible $\mathfrak{g}$-module of highest weight λ, and set $V^* = V(\lambda^*)$. Let K be a $\mathfrak{g}$-submodule of $V \wedge V$. Then, the following conditions are equivalent:*

(1) $\mathcal{R}(V, K) = \{0\}$.
(2) $\dim_{\mathbb{C}} \mathcal{W}(V, K) < \infty$.
(3) $2\lambda^* - \beta \notin \mathrm{Weight}^+(K^\perp)$, *for all* $\beta \in \Delta$ *such that* $(\beta, \lambda^*) \neq 0$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 2.4. The implication (3) $\Rightarrow$ (1) follows from Theorem 4.6 and Lemma 3.3.

For the implication (1) $\Rightarrow$ (3), set $\mu := 2\lambda^* - \beta$, and suppose $\mu \in \mathrm{Weight}^+(K^\perp)$, for some $\beta \in \Delta$ such that $(\beta, \lambda^*) \neq 0$. In this case, there is a submodule $W \subseteq K^\perp$ isomorphic to $V(\mu)$. On the other hand, the submodule $K(\lambda^*, \beta) \subseteq V^* \wedge V^*$ is also isomorphic to $V(\mu)$, by Lemma 3.4. By the uniqueness property from [1, Ch. VIII, §7, Exercice 17], $K(\lambda^*, \beta) = W$. Hence, by Proposition 4.1, v_{λ^*} is a non-zero vector in $\mathcal{R}(V, K)$, and we are done. □

5. Weyman modules

In this section, we treat in detail the case when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The representation theory of this simple Lie algebra is of course classical; for a thorough treatment, we refer to Fulton and Harris [7].

The dual Cartan subalgebra, $\mathfrak{h}^*$, is spanned by functionals t_1 and t_2 (the dual coordinates on the subspace of diagonal 2×2 complex matrices), subject to the single relation $t_1 + t_2 = 0$. The set $\Phi^+ = \Delta$ consists of a single simple root, $\beta = t_1 - t_2$. The defining (2-dimensional) representation is $V(\lambda_1)$, where $\lambda_1 = t_1$. Moreover, all the finite-dimensional, irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ are of the form

$$V_n = V(n\lambda_1) = \mathrm{Sym}_n(V(\lambda_1)), \tag{28}$$

for some $n \geq 0$. Clearly, $\dim V_n = n + 1$. Furthermore, $V_n^* = V_n$; in other words, all irreps of $\mathfrak{sl}_2(\mathbb{C})$ are self-dual.

For each $n \geq 1$, the second exterior power of V_n decomposes into irreducibles, according to the Clebsch-Gordan rule:

$$V_n \wedge V_n = \bigoplus_{j \geq 0} V_{2n-2-4j}. \tag{29}$$

In particular, all summands in (29) occur with multiplicity 1, and V_{2n-2} is always one of those direct summands.

Proposition 5.1. *Let K be an $\mathfrak{sl}_2(\mathbb{C})$-submodule of $V_n \wedge V_n$. The following are equivalent:*

(1) *The variety $\mathcal{R}(V_n, K)$ consists only of $0 \in V_n^*$.*
(2) *The $\mathbb{C}$-vector space $\mathcal{W}(V_n, K)$ is finite-dimensional.*
(3) *The representation K contains V_{2n-2} as a direct summand.*

Proof. Let us apply Corollary 4.7 to this situation. We have $\lambda^* = \lambda = nt_1$ and $\beta = t_1 - t_2$; thus, $(\lambda^*, \beta) \neq 0$ and $2\lambda^* - \beta = (2n-2)\lambda_1$. The desired conclusion follows at once. □

Following Eisenbud [5], let us single out an important particular case of the above construction.

Definition 5.2. For each $n \geq 1$, the corresponding *Weyman module* is $W(n) = \mathcal{W}(V_n, V_{2n-2})$, viewed as a graded module over the polynomial ring $\mathrm{Sym}(V_n)$.

Note that V_{2n-2} belongs to the set $U_{n+1,2(n+1)-3}$ from (8), and thus, it is in the critical range identified in Proposition 2.7.

The first part of the next corollary recovers an assertion from [5].

Corollary 5.3. *For each $n \geq 1$, the following hold.*

(1) *The Weyman module $W(n) = \mathcal{W}(V_n, V_{2n-2})$ has finite dimension as a $\mathbb{C}$-vector space.*
(2) *Given an $\mathfrak{sl}_2(\mathbb{C})$-submodule $K \subset V_n \wedge V_n$, the corresponding Koszul module $\mathcal{W}(V_n, K)$ is finite-dimensional over $\mathbb{C}$ if and only if $\mathcal{W}(V_n, K)$ is a quotient of $W(n)$.*

As explained by Eisenbud in [5], Weyman showed that the vanishing of $W_{n-2}(n)$, for all $n \geq 1$, implies the generic Green Conjecture on free resolutions of canonical curves.

6. Alexander invariants and resonance varieties of groups

6.1. **The Alexander invariant of a group.** Let G be a group, and let $(x, y) = xyx^{-1}y^{-1}$ denote the group commutator. The derived subgroup, $G' = (G, G)$, is a normal subgroup; the quotient group, $G_{\mathrm{ab}} = G/G'$, is the maximal abelian quotient of G. Also let $G'' = (G', G')$ be the second derived subgroup; then G/G'' is the maximal metabelian quotient of G.

The abelianization map, $\mathrm{ab}\colon G \twoheadrightarrow G_{\mathrm{ab}}$, factors through G/G'', yielding an exact sequence, $0 \to G'/G'' \to G/G'' \to G_{\mathrm{ab}} \to 0$. Conjugation in G/G'' naturally makes the abelian group G'/G'' into a module over the group ring $\mathbb{Z}[G_{\mathrm{ab}}]$. Following W. Massey [13], we call the complexification of this module,

$$B(G) := (G'/G'') \otimes \mathbb{C} = H_1(G', \mathbb{C}), \tag{30}$$

the *Alexander invariant* of G. By construction, $B(G)$ is a module over the group algebra $R = \mathbb{C}[G_{\mathrm{ab}}]$, with module structure given by $\bar{h} \cdot \bar{g} = \overline{hgh^{-1}}$, for elements $\bar{h} \in G/G'$, represented by $h \in G$, and $\bar{g} \in G'/G''$, represented by $g \in G'$.

As a slight variation on this construction, let $\mathrm{abf}\colon G \to G_{\mathrm{abf}}$ be the projection to the maximal torsion-free abelian quotient, $G_{\mathrm{abf}} = G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}})$, and define the *reduced Alexander invariant* to be the vector space

$$\widetilde{B}(G) = H_1(\ker(\mathrm{abf}), \mathbb{C}), \tag{31}$$

viewed as a module over the group algebra $\widetilde{R} = \mathbb{C}[G_{\mathrm{abf}}]$. We then have an epimorphism $B(G) \to \widetilde{B}(G)$, equivariant with respect to the canonical projection $R \to \widetilde{R}$.

6.2. **Infinitesimal Alexander invariant.** For the rest of this section, we will assume G is a finitely generated group. Set $V = H_1(G, \mathbb{C})$, and identify the dual vector space, V^*, with $H^1(G, \mathbb{C})$. Let $\cup_G\colon V^* \wedge V^* \to H^2(G, \mathbb{C})$ be the cup-product map. Its kernel, $K^\perp \subset V^* \wedge V^*$, is the orthogonal complement to a linear subspace $K \subset V \wedge V$; put another way, $K = \operatorname{im}(\partial_G)$, where $\partial_G\colon H_2(G, \mathbb{C}) \to V \wedge V$ is the transpose of $\cup_G$.

Let us define the *infinitesimal Alexander invariant* of G to be the S-module

$$\mathfrak{B}(G) = \mathcal{W}(V, K), \tag{32}$$

where $S = \operatorname{Sym}(V)$. It is readily seen that this definition agrees with the one from [14], modulo a degree shift by 2.

Now let I be the augmentation ideal of the ring $R = \mathbb{C}[G_{\mathrm{ab}}]$. The powers of I define a filtration on the Alexander invariant of G; the associated graded object,

$$\operatorname{gr} B(G) = \bigoplus_{q \geq 0} I^q B(G)/I^{q+1} B(G), \tag{33}$$

may be viewed as a graded module over the ring $\operatorname{gr} R = S$. Similar considerations apply to the S-module $\operatorname{gr} \widetilde{B}(G)$.

Work from [13, 14, 16] as well as [2, Proposition 2.4], implies that

$$\dim_{\mathbb{C}} \operatorname{gr}_q B(G) = \dim_{\mathbb{C}} \operatorname{gr}_q \widetilde{B}(G) \leq \dim_{\mathbb{C}} \mathfrak{B}_q(G), \tag{34}$$

for all $q \geq 0$.

6.3. **The formal situation.** Now suppose the (finitely generated) group G is 1-*formal*, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is the completion of a quadratic Lie algebra (we refer to [14] for more details on this notion). Then, it follows from [4] that

$$\operatorname{gr} B(G) \cong \mathfrak{B}(G), \tag{35}$$

as graded vector spaces.

Recall that an R-module M is said to be nilpotent if $I^q \cdot M = 0$, for some $q \geq 1$. Clearly, if M is nilpotent, then $M = \operatorname{gr} M$, as vector spaces. The following corollary is now immediate.

Corollary 6.1. *Suppose G is a 1-formal group. Then:*

1. $\operatorname{gr} B(G) = \operatorname{gr} \widetilde{B}(G) = \mathfrak{B}(G)$, *as graded vector spaces.*
2. *If $B(G)$ is nilpotent, then* $\dim_{\mathbb{C}} B(G) = \dim_{\mathbb{C}} \mathfrak{B}(G) < \infty$.
3. *If $\widetilde{B}(G)$ is nilpotent, then* $\dim_{\mathbb{C}} \widetilde{B}(G) = \dim_{\mathbb{C}} \mathfrak{B}(G) < \infty$.

6.4. **Resonance varieties of groups.** As before, let G be a finitely generated group, with $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G) \subset V^* \wedge V^*$. The resonance variety of G is then defined as

$$\mathcal{R}(G) = \mathcal{R}(V, K). \tag{36}$$

This definition coincides (at least away from the origin) with the usual definition of the first resonance variety, $\mathcal{R}^1_1(G, \mathbb{C})$.

Proposition 6.2. *Let V be an n-dimensional $\mathbb{C}$-vector space.*

1. *For any linear subspace $K \subseteq V \wedge V$ defined over $\mathbb{Q}$, there is a finitely presented, commutator-relators group G with $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G)$.*
2. *If $m \geq 2n - 3$, the Zariski open set $U_{n,m} = \{K \in \operatorname{Gr}_m(V \wedge V) \mid \mathcal{R}(V, K) = \{0\}\}$ contains a rational m-plane.*

Proof. To prove (1), fix a basis $e_1, \dots, e_n$ for V, and pick a basis $v_1, \dots, v_m$ for K, with entries $v_k = \sum_{1 \leq i < j \leq n} c^k_{i,j} e_i \wedge e_j$ having integral coefficients. Consider the group with presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, where $r_k = \prod_{1 \leq i < j \leq n} (x_i, x_j)^{c^k_{i,j}}$. A standard Fox calculus computation shows that $K = \operatorname{im}(\partial_G)$.

To prove (2), note that the rational points in $\mathbb{C}^{m\binom{n}{2}}$ form a Zariski-dense subset. Hence, the conclusion follows from Proposition 2.7. □

As an immediate corollary, we see that the bound from Proposition 2.7 is sharp, even for m-planes coming from finitely presented groups.

Corollary 6.3. *For each $n \geq 2$, there is a finitely presented group G such that $b_1(G) = n$, $\operatorname{codim}(\ker(\cup_G)) = 2n-3$, and $\mathcal{R}(G) = \{0\}$.*

Example 6.4. For $n = 4$, consider the group

$$G = \langle x_1, \dots, x_4 \mid (x_1, x_2),\ (x_2, x_3),\ (x_3, x_4),\ (x_1, x_4),\ (x_1, x_3) \cdot (x_2, x_4) \rangle.$$

In Plücker coordinates, the Grassmannian $\operatorname{Gr}_2(\mathbb{C}^4) \subset \mathbb{P}(\mathbb{C}^6)$ is given by the equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$, while the plane $\mathbb{P}(K^\perp) = \mathbb{P}(\ker(\cup_G)) \in \mathbb{P}(\mathbb{C}^6)$ is given by $p_{12} = p_{23} = p_{34} = p_{14} = p_{13} + p_{24} = 0$. Clearly, $\mathbb{P}(K^\perp) \cap \operatorname{Gr}_2(\mathbb{C}^4) = \emptyset$; thus, by Lemma 2.5, $\mathcal{R}(G) = \{0\}$.

Finally, recall that the *deficiency* of a finitely presented group G, written $\operatorname{def}(G)$, is the supremum of the difference between the number of generators and the number of relators, taken over all finite presentations of G.

Corollary 6.5. *Let G be a finitely presented group. Suppose $\mathcal{R}(G) = \{0\}$. Then $\operatorname{def}(G) \leq 3 - b_1(G)$.*

Proof. Let $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_q \rangle$ be a finite presentation. Set $n = b_1(G)$ and $k = \dim(\operatorname{im}(\partial_G))$. Then $k \leq b_2(G) \leq q - m + n$. From the hypothesis and Proposition 2.7, we have that $2n - 3 \leq k$. Thus, $2n - 3 \leq q - m + n$, and so $m - q \leq 3 - n$. The conclusion follows. □

7. Torelli groups and vanishing resonance

7.1. **Torelli groups of free groups.** Let F_n be the free group on n generators. Identify the group $H = (F_n)_{\mathrm{ab}}$ with $\mathbb{Z}^n$ and the group $\operatorname{Aut}(\mathbb{Z}^n)$ with $\operatorname{GL}_n(\mathbb{Z})$. As is well-known, the Torelli group, $\operatorname{OA}_n = \mathcal{T}_{F_n}$, is finitely generated. Furthermore, the canonical homomorphism $\operatorname{Out}(F_n) \to \operatorname{GL}_n(\mathbb{Z})$ is surjective; thus, there is a natural $\operatorname{GL}_n(\mathbb{Z})$-action on the homology groups of OA_n.

Work of Andreadakis, Cohen–Pakianathan, Farb and Kawazumi (see [17]) shows that the action by restriction of $\operatorname{SL}_n(\mathbb{Z})$ on $V = H_1(\operatorname{OA}_n, \mathbb{C})$ extends to a rational, irreducible representation of the simply-connected, complex, semisimple, linear algebraic group $\operatorname{SL}_n(\mathbb{C})$, and thus, of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, which can be explicitly identified.

Following Fulton and Harris [7], denote by $t_1, \dots, t_n$ the dual coordinates of the diagonal matrices from $\mathfrak{gl}_n(\mathbb{C})$, and set $\lambda_i = t_1 + \cdots + t_i$, for $1 \leq i \leq n$. Let $\mathfrak{h}_n$ be the (diagonal) Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$; then $\mathfrak{h}_n^* = \mathbb{C}\text{-span}\,\{t_1, \dots, t_n\}/\mathbb{C} \cdot \lambda_n$. The standard set of positive roots is $\Phi^+ = \{t_i - t_j \mid 1 \leq i < j \leq n\}$, while the set of simple roots is $\Delta = \{t_i - t_{i+1} \mid 1 \leq i \leq n-1\}$. The finite-dimensional irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ are parametrized by tuples $(a_1, \dots, a_{n-1})$ of non-negative integers; to such a tuple, there corresponds an irreducible representation $V(\lambda)$, with highest weight $\lambda = \sum_{i=1}^{n-1} a_i \lambda_i$.

The following theorem recovers a result from [16].

Theorem 7.1. *For each $n \geq 4$, the resonance variety $\mathcal{R}(\operatorname{OA}_n)$ vanishes.*

Proof. Let $V^* = H^1(\operatorname{OA}_n, \mathbb{C})$, and let $K^\perp \subset V^* \wedge V^*$ be the kernel of the cup-product map in degree 1. Set $\lambda = \lambda_2 + \lambda_{n-1}$, so that $\lambda^* = \lambda_1 + \lambda_{n-2}$, and also set $\mu = \lambda_1 + \lambda_{n-2} + \lambda_{n-1}$. As shown by Pettet in [17], $V^* = V(\lambda^*)$ and $K^\perp = V(\mu)$, as $\mathfrak{sl}_n(\mathbb{C})$-modules.

It is immediate to see that $2\lambda^* - \mu = t_1 - t_{n-1}$ is not a simple root. It follows from Theorem 4.6 that $\mathcal{R}(V, K) = \{0\}$. □

Remark 7.2. When $n = 3$, the above proof breaks down, since $t_1 - t_2$ is a simple root. In fact, $K^\perp = V^* \wedge V^*$ in this case, and so $\mathcal{R}(V, K) = V^*$.

Corollary 7.3. *For each $n \geq 4$, let $V = V(\lambda_2+\lambda_{n-1})$ and let $K^\perp = V(\lambda_1+\lambda_{n-2}+\lambda_{n-1}) \subset V^*\wedge V^*$ be the Pettet summand, as above. Then $\dim \mathcal{W}(V, K) < \infty$ and $\dim \operatorname{gr}_q B(\mathrm{OA}_n) \leq \dim \mathcal{W}_q(V, K)$, for all $q \geq 0$.*

Proof. Follows from Lemma 2.4 and inequality (34). □

7.2. **Torelli groups of surfaces.** Let Σ_g be a Riemann surface of genus g, and let $\mathcal{T}_g = \mathcal{T}_{\pi_1(\Sigma_g)}$ be the associated Torelli group. For $g \leq 1$, the group $\mathcal{T}_g$ is trivial, while for $g = 2$, it is not finitely generated. So we will assume from now on that $g \geq 3$, in which case it is known that $\mathcal{T}_g$ is finitely generated.

The canonical homomorphism $\mathrm{Out}^+(\pi_1(\Sigma_g)) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is surjective; thus, there is a natural $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on the homology groups of $\mathcal{T}_g$. Work of D. Johnson [12] shows that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $V = H_1(\mathcal{T}_g, \mathbb{C})$ extends to a rational, irreducible representation of the simply-connected, complex, semisimple, linear algebraic group $\mathrm{Sp}_{2g}(\mathbb{C})$, and thus, of the Lie algebra $\mathfrak{sp}_{2g}(\mathbb{C})$.

Let $\mathfrak{h} \subset \mathfrak{sp}_{2g}(\mathbb{C})$ be the Cartan subalgebra of diagonal matrices, and let $t_1, \dots, t_g$ be the standard basis of $\mathfrak{h}^*$, orthogonal with respect to the dual Killing form. As in [7], we take the set of simple roots to be $\Delta = \{t_1 - t_2, t_2 - t_3, \dots, t_{g-1} - t_g, 2t_g\}$. The fundamental dominant weights are $\lambda_i = t_1 + \cdots + t_i$, with $1 \leq i \leq g$. The finite-dimensional irreducible representations of $\mathfrak{sp}_{2g}(\mathbb{C})$ are parametrized by tuples $(a_1, \dots, a_g)$ of non-negative integers; to such a tuple, there corresponds an irreducible representation $V(\lambda)$, with highest weight $\lambda = \sum_{i=1}^g a_i\lambda_i$.

The following theorem recovers a result from [3].

Theorem 7.4. *For each $g \geq 4$, the resonance variety $\mathcal{R}(\mathcal{T}_g)$ vanishes and $\dim \mathcal{W}(V, K) < \infty$.*

Proof. Let $V^* = H^1(\mathcal{T}_g, \mathbb{C})$, and let $K^\perp \subset V^* \wedge V^*$ be the kernel of the cup-product map in degree 1. Work of Hain [8] identifies these $\mathfrak{sp}_{2g}(\mathbb{C})$-representation spaces, as follows: $V^* = V(\lambda_3)$ and $K^\perp = V(2\lambda_2) \oplus V(0)$. Moreover, the decomposition of $V^* \wedge V^*$ into irreducibles is multiplicity-free.

Let us apply Corollary 4.7 to this situation. The only simple root $\beta \in \Delta$ such that $(\lambda_3, \beta) \neq 0$ is $\beta = t_3 - t_4$. Clearly, $2\lambda_3 - \beta = \lambda_2 + \lambda_4$ does not belong to $\mathrm{Weight}^+(K^\perp)$. Hence, $\mathcal{R}(V, K) = \{0\}$. By Lemma 2.4, $\dim \mathcal{W}(V, K) < \infty$. □

Remark 7.5. When $g = 3$, the above proof breaks down. Indeed, take $\beta = 2t_3 \in \Delta$; then $(\lambda^*, \beta) \neq 0$ and $2\lambda^* - \beta = 2\lambda_2$ belongs to $\mathrm{Weight}^+(K^\perp)$. Hence, $\mathcal{R}(V, K) \neq \{0\}$. In fact, $K^\perp = V^* \wedge V^*$ in this case, and thus $\mathcal{R}(V, K) = V^*$.

Corollary 7.6. *For each $g \geq 6$, let $V = V(\lambda_3)$ and $K^\perp = V(2\lambda_2) \oplus V(0)$. Then $\dim \widetilde{B}(\mathcal{T}_g) = \dim \mathcal{W}(V, K)$.*

Proof. As shown in [8], the group $\mathcal{T}_g$ is 1-formal, as long as $g \geq 6$. Furthermore, as shown in [2], the $\widetilde{R}$-module $\widetilde{B}(\mathcal{T}_g)$ is nilpotent, provided $g \geq 4$. The conclusion follows from Corollary 6.1(3), and the fact that all representations of $\mathfrak{sp}_{2g}(\mathbb{C})$ are self-dual. □

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