

The pure braid groups and their relatives

Alexander I. Suciú and He Wang

Abstract In this mostly survey paper, we investigate the resonance varieties, the lower central series ranks, and the Chen ranks, as well as the residual and formality properties of several families of braid-like groups: the pure braid groups P_n , the welded pure braid groups wP_n , the virtual pure braid groups vP_n , as well as their ‘upper’ variants, wP_n^+ and vP_n^+ . We also discuss several natural homomorphisms between these groups, and various ways to distinguish among the pure braid groups and their relatives.

Key words: Pure braid groups, welded pure braid group, virtual pure braid groups, lower central series, Chen ranks, resonance varieties, residually nilpotent, formality.

1 Introduction

1.1 Cast of characters

Let F_n be the free group on generators x_1, \dots, x_n , and let $\text{Aut}(F_n)$ be its automorphism group. Magnus [51] showed that the map $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on the abelianization $(F_n)^{\text{ab}} = \mathbb{Z}^n$ is surjective. Furthermore, the kernel of this homomorphism, denoted by IA_n , is generated by automorphisms α_{ij} and α_{ijk} ($1 \leq i \neq j \neq k \leq n$) which send x_i to $x_j x_i x_j^{-1}$ and

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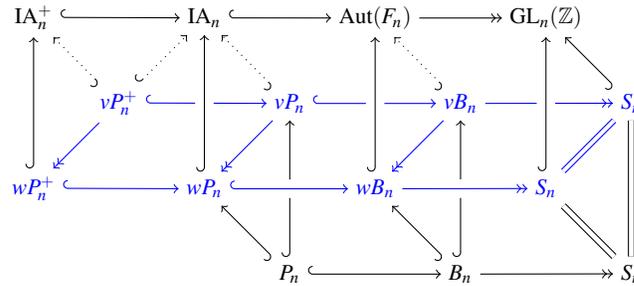


Fig. 1 Braid-like groups and automorphism groups of free groups.

$x_i x_j x_k x_j^{-1} x_k^{-1}$, respectively, and leave invariant the remaining generators of F_n . The subgroup generated by the automorphisms α_{ij} and α_{ijk} with $i < j < k$ is denoted by IA_n^+ .

An automorphism of F_n is called a ‘permutation-conjugacy’ if it sends each generator x_i to a conjugate of $x_{\tau(i)}$, for some permutation $\tau \in S_n$. The classical Artin braid group B_n is the subgroup of $\text{Aut}(F_n)$ consisting of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$, see for instance Birman’s book [17]. The kernel of the canonical projection from B_n to the symmetric group S_n is the pure braid group P_n on n strings. As shown by Fadell, Fox, and Neuwirth [33, 38], a classifying space for P_n is $\text{Conf}_n(\mathbb{C})$, the configuration space of n ordered points on the complex line.

The set of all permutation-conjugacy automorphisms of F_n forms a subgroup of $\text{Aut}(F_n)$, denoted by $B\Sigma_n$. The subgroup $P\Sigma_n = B\Sigma_n \cap \text{IA}_n$ is generated by the Magnus automorphisms α_{ij} ($1 \leq i \neq j \leq n$), while the subgroup $P\Sigma_n^+ = P\Sigma_n \cap \text{IA}_n^+$ is generated by the automorphisms α_{ij} with $i < j$. In [56], McCool gave presentations for the groups $P\Sigma_n$ and $P\Sigma_n^+$; these groups are now also called the McCool groups and the upper McCool groups, respectively.

The *welded braid groups* were introduced by Fenn, Rimányi, and Rourke in [37], who showed that the welded braid group wB_n is isomorphic to $B\Sigma_n$. These groups, together with the *welded pure braid groups* $wP_n \cong P\Sigma_n$ and the *upper welded pure braid groups* $wP_n^+ \cong P\Sigma_n^+$ have generated quite a bit of interest since then, see for instance [3, 4, 8, 15, 30] and references therein. The welded pure braid group wP_n can be identified with group of motions of n unknotted, unlinked circles in the 3-sphere. As shown by Brendle and Hatcher in [18], this group can be realized as the fundamental group of the space of configurations of parallel rings in \mathbb{R}^3 .

A related class of groups are the *virtual braid groups* vB_n , which were introduced by Kauffman in [45] in the context of virtual knot theory, see also [42]. The kernel of the canonical epimorphism $vB_n \rightarrow S_n$ is called the *virtual pure braid group* vP_n . In [5], Bardakov found a concise presentation for vP_n , and defined accordingly the

upper virtual pure braid group νP_n^+ . Whether or not the virtual (pure) braid groups are subgroups of $\text{Aut}(F_n)$ is an open question that goes back to [5].

The groups νP_n and νP_n^+ were also independently studied by Bartholdi, Enriquez, Etingof, and Rains [11] and P. Lee [50] as groups arising from the Yang-Baxter equations. Classifying spaces for these groups (also known as the quasi-triangular groups and the triangular groups, respectively) can be constructed by taking quotients of permutahedra by suitable actions of the symmetric groups.

The groups mentioned so far fit into the diagram from Figure 1 (a related diagram can be found in [3]). We will discuss presentations for these groups, various extensions and homomorphisms between them, as well as their centers in §2.

1.2 Lie algebras, LCS ranks, and formality

To any finitely generated group G , there corresponds a graded Lie algebra, $\text{gr}(G)$, obtained by taking the direct sum of the successive quotients of the lower central series of G , and tensoring with \mathbb{C} . The *LCS ranks* of the group G are defined as the dimensions, $\phi_k(G) = \dim \text{gr}_k(G)$, of the graded pieces of this Lie algebra. As explained in Theorem 9, the computation of these ranks is greatly simplified if the group G satisfies certain formality properties, and its cohomology algebra is Koszul.

The set of primitive elements of the completed group algebra $\widehat{\mathbb{C}G}$ is a complete, filtered Lie algebra over \mathbb{C} , called the *Malcev Lie algebra* of G , and denoted by $\mathfrak{m}(G)$. By a theorem of Quillen [62], there exists an isomorphism of graded Lie algebras between $\text{gr}(G)$ and $\text{gr}(\mathfrak{m}(G))$.

The group G is said to be *graded-formal* if its associated graded Lie algebra, $\text{gr}(G)$, admits a quadratic presentation. The group G is said to be *filtered-formal* if there exists an isomorphism of filtered Lie algebras between $\mathfrak{m}(G)$ and the degree completion of $\text{gr}(G)$. Furthermore, the group G is called *1-formal* if it is graded-formal and filtered-formal, or, equivalently, if there is a 1-quasi-isomorphism between the 1-minimal model of G and the cohomology algebra $H^*(G, \mathbb{C})$ endowed with the zero differential. We refer to [69] for a comprehensive study of these formality notions for groups.

A presentation for the Malcev Lie algebra of P_n was given by Kohno in [48], while the associated graded Lie algebra $\text{gr}(P_n)$ and its graded ranks were computed by Kohno [49] and Falk–Randell [35]. It was also realized around that time that the pure braid groups P_n are 1-formal. As shown by Berceanu and Papadima in [15], the Malcev Lie algebras of wP_n and wP_n^+ admit quadratic presentations, that is, the groups wP_n and wP_n^+ are 1-formal. Furthermore, as shown in [11, 50], the groups νP_n and νP_n^+ are graded-formal. On the other hand, we show in [71] that the virtual pure braid groups νP_n and νP_n^+ are 1-formal if and only if $n \leq 3$.

A lot is also known about the residual properties of the pure braid-like groups, especially as they relate to the lower central series. For instance, a theorem of Berceanu and Papadima [15], which uses work of Andreadakis [1] and an idea of Hain [43], shows that the groups IA_n are residually torsion-free nilpotent, for all n .

Table 1 Hilbert series, Koszulness, and formality of pure braid-like groups.

G	Hilbert series $\text{Hilb}(H^*(G; \mathbb{C}), t)$	Koszulness	1-Formality
P_n	$\prod_{j=1}^{n-1} (1 + jt)$ [49]	Yes [2, 49, 64]	Yes [48]
wP_n	$(1 + nt)^{n-1}$ [44]	No (for $n \geq 4$) [29]	Yes [15]
wP_n^+	$\prod_{j=1}^{n-1} (1 + jt)$ [28]	Yes [22]	Yes [15]
νP_n	$\sum_{i=0}^{n-1} \binom{n-1}{i} \frac{n!}{(n-i)!} t^i$ [11]	Yes [11, 50]	No (for $n \geq 4$) [70]
νP_n^+	$\sum_{j=1}^n \frac{\left(\sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (k-i)^n \right) t^{n-j}}{j!}$ [11]	Yes [11, 50]	No (for $n \geq 4$) [70]

Thus, the groups P_n , wP_n , and wP_n^+ also enjoy this property. The fact that the pure braid groups P_n are residually torsion-free nilpotent also follows from the work of Falk and Randell [35, 36]. It is also known that the virtual pure braid groups νP_n and νP_n^+ are residually torsion-free nilpotent for $n \leq 3$, but it is not known whether this is the case for $n \geq 4$.

1.3 Resonance varieties and Chen ranks

We conclude our survey with a discussion of the cohomology algebras of the pure braid-like groups, and of two other related objects: the resonance varieties attached to these graded algebras, and the Chen ranks associated to the groups themselves.

The cohomology algebra of the classical pure braid group P_n was computed by Arnol'd in his seminal paper on the subject, [2]. An explicit presentation for the cohomology algebra of the McCool group wP_n was given by Jensen, McCammond and Meier [44], thereby confirming a conjecture of A. Brownstein and R. Lee. Using different methods, F. Cohen, Pakianathan, Vershinin, and Wu [28] determined the cohomology algebra of the upper McCool group wP_n^+ . Finally, the cohomology algebras of the virtual pure braid groups νP_n and νP_n^+ were computed by Bartholdi et al. [11] and Lee [50].

For all these groups G , the cohomology algebra $A = H^*(G, \mathbb{C})$ is quadratic, i.e., it is generated in degree 1 and the ideal of relations is generated in degree 2. In fact, for all but the groups wP_n , $n \geq 4$, the ideal of relations admits a quadratic Grbner

Table 2 Resonance and Chen ranks of braid-like groups.

G	First resonance variety $\mathcal{R}_1(G) \subseteq H^1(G; \mathbb{C})$	Chen ranks $\theta_k(G)$, $k \geq 3$	Resonance– Chen ranks formula
P_n	$\binom{n}{3} + \binom{n}{4}$ planes [27]	$(k-1)\binom{n+1}{4}$ [24]	Yes [27]
wP_n	$\binom{n}{2}$ planes and $\binom{n}{4}$ linear spaces of dimension 3 [21]	$(k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$ for $k \gg 3$ [23]	Yes [23]
wP_n^+	$(n-i)$ linear spaces of dimension $i \geq 2$ [71]	$\sum_{i=3}^k \binom{n+i-2}{i+1} + \binom{n+1}{4}$ [71]	No [71]
νP_3	$H^1(\nu P_3, \mathbb{C}) = \mathbb{C}^6$ [10, 70]	$\binom{k+3}{5} + \binom{k+2}{4} + \binom{k+1}{3} + 6\binom{k}{2} + k - 2$ [70]	No [70]
νP_4^+	3-dimensional non-linear subvariety of degree 6 [70]	$(k^3 - 1) + \binom{k}{2}$ [70]	No [70]

basis, and so the algebra A is Koszul. For more details and references regarding this topic, we direct the reader to Table 1 and to §3.1.

Given a group G satisfying appropriate finiteness conditions, the resonance varieties $\mathcal{R}_s^i(G)$ are certain closed, homogeneous subvarieties of the affine space $A^1 = H^1(G; \mathbb{C})$, defined by means of the vanishing cup products in the cohomology algebra $A = H^*(G, \mathbb{C})$. We restrict our attention here to the first resonance variety $\mathcal{R}_1(G) = \mathcal{R}_1^1(G)$ attached to a finitely generated group G . This variety consists of all elements $a \in A^1$ for which there exists an element $b \in A^1$ such that $a \cup b = 0$, yet b is not proportional to a .

The aforementioned computations of the cohomology algebras of the various pure braid-like groups allows one to determine the corresponding resonance varieties, at least in principle. In the case of the first resonance varieties of the groups P_n , wP_n , and wP_n^+ , complete answers can be found in [27], [21], and [71], respectively, while for the virtual pure braid groups, partial answers are given in [70]. We list some of the features of these varieties in Table 2.

By comparing the resonance varieties of the groups P_n and wP_n^+ , it can be shown that these groups are not isomorphic for $n \geq 4$ (cf. [71]); this answers a question of F. Cohen et al. [28], see Remark 18. By computing the resonance variety $\mathcal{R}_1(\nu P_4^+)$, and using the Tangent Cone Theorem from [32], we prove that the group νP_4^+ is not 1-formal. In view of the retraction property for 1-formality established in [69], we conclude that the groups νP_n and νP_n^+ are not 1-formal for $n \geq 4$.

The *Chen ranks* of a finitely generated group G are the dimensions, $\theta_k(G) = \dim \text{gr}_k(G/G'')$, of the graded pieces of the graded Lie algebra associated to the maximal metabelian quotient of G . In [19], K.-T. Chen computed the Chen ranks

of the free groups F_n , while in [54], W.S. Massey gave an alternative method for computing the Chen ranks of a group G in terms of the Alexander invariant G'/G'' .

The Chen ranks of the pure braid groups P_n were computed in [24], while an explicit relation between the Chen ranks and the resonance varieties of an arrangement group was conjectured in [67]. Building on work from [26, 63, 65] and especially [58], Cohen and Schenck confirmed this conjecture in [23] for a class of 1-formal groups which includes arrangement groups. In the process, they also computed the Chen ranks $\theta_k(wP_n)$ for k sufficiently large.

Using the Grbner basis algorithm from [24, 26], we compute in [71] all the Chen ranks of the upper McCool groups wP_n^+ . This computation, recorded here in Theorem 16, shows that, for each $n \geq 4$, the group wP_n^+ is not isomorphic to either the pure braid group P_n , or to the product $\prod_{i=1}^{n-1} F_i$, although these three groups share the same LCS ranks and the same Betti numbers. We also provide the Chen ranks of the groups vP_n and vP_n^+ for low values of n . The full computation of the Chen ranks of the virtual pure braid groups remains to be done.

2 Braid groups and their relatives

2.1 Braid groups and pure braid groups

Let $\text{Aut}(F_n)$ be the group of (right) automorphisms of the free group F_n on generators x_1, \dots, x_n . Recall that the Artin braid group B_n consists of those permutation-conjugacy automorphisms which fix the word $x_1 \cdots x_n \in F_n$. In particular, $B_1 = \{1\}$ and $B_2 = \mathbb{Z}$. The natural inclusion $\alpha_n: B_n \hookrightarrow \text{Aut}(F_n)$ is also known as the Artin representation of the braid group.

For each $1 \leq i < n$, let σ_i be the braid automorphism which sends x_i to $x_i x_{i+1} x_i^{-1}$ and x_{i+1} to x_i , while leaving the other generators of F_n fixed. As shown for instance in [17], the braid group B_n is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$, subject to the well-known relations

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i, & |i-j| \geq 2. \end{cases} \quad (\text{R1})$$

On the other hand, the symmetric group S_n has a presentation with generators s_i for $1 \leq i \leq n-1$ and relations

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n-2, \\ s_i s_j = s_j s_i, & |i-j| \geq 2, \\ s_i^2 = 1, & 1 \leq i \leq n-1; \end{cases} \quad (\text{R2})$$

The canonical projection from the braid group to the symmetric group, which sends the elementary braid σ_i to the transposition s_i , has kernel the *pure braid group*

on n strings,

$$P_n = \ker(\phi: B_n \rightarrow S_n) = B_n \cap \text{IA}_n, \quad (1)$$

where $\phi(\sigma_i) = s_i$ for $1 \leq i \leq n-1$. The group P_n is generated by the n -stranded braids

$$A_{ij} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \quad (2)$$

for $1 \leq i < j \leq n$. It is readily seen that $P_1 = \{1\}$, $P_2 = \mathbb{Z}$, and $P_3 \cong F_2 \times \mathbb{Z}$. More generally, as shown by Fadell and Neuwirth [33] (see also [35, 36, 25]), the pure braid group P_n can be decomposed as an iterated semi-direct product of free groups,

$$P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1, \quad (3)$$

where $\alpha_{n-1}: P_{n-1} \hookrightarrow \text{Aut}(F_n)$ is the restriction of the Artin representation of the braid group B_{n-1} to the pure braid subgroup P_{n-1} .

Work of Chow [20] and Birman [17] shows that the center $Z(P_n)$ of the pure braid group on $n \geq 2$ strands is infinite cyclic, generated by the full twist braid $\prod_{1 \leq i < j \leq n} A_{ij}$. It follows that $P_n \cong \bar{P}_n \times \mathbb{Z}$, where $\bar{P}_n = P_n/Z(P_n)$.

2.2 Welded braid groups

The set of all permutation-conjugacy automorphisms of the free group of rank n forms the braid-permutation group wB_n . This group has a presentation with generators σ_i and s_i ($1 \leq i < n$) and relations (R1) and (R2), as well as

$$\begin{cases} s_i \sigma_j = \sigma_j s_i, & |i-j| \geq 2, \\ \sigma_i s_{i+1} s_i = s_{i+1} s_i \sigma_{i+1}, & 1 \leq i \leq n-2, \end{cases} \quad (\text{R3})$$

and

$$s_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i s_{i+1}, \quad 1 \leq i \leq n-2. \quad (\text{R4})$$

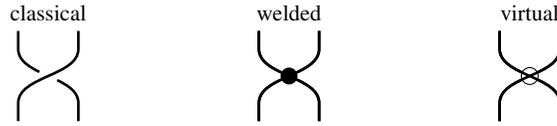


Fig. 2 Braid crossings.

The three types of braid crossings mentioned above are depicted in Figure 2.

The welded pure braid group wP_n , also known as the group of basis-conjugating automorphisms in [4, 37, 21], or the McCool group in [15], is defined as

$$wP_n = \ker(\rho: wB_n \rightarrow S_n) = wB_n \cap \text{IA}_n, \quad (4)$$

where $\rho(\sigma_i) = \rho(s_i) = s_i$ for $1 \leq i \leq n-1$. As shown by McCool in [56], this group is generated by the Magnus automorphisms α_{ij} , for all $1 \leq i \neq j \leq n$, subject to the relations

$$\begin{aligned} \alpha_{ij}\alpha_{ik}\alpha_{jk} &= \alpha_{jk}\alpha_{ik}\alpha_{ij}, & \text{for } i, j, k \text{ distinct,} \\ [\alpha_{ij}, \alpha_{st}] &= 1, & \text{if } \{i, j\} \cap \{s, t\} = \emptyset, \\ [\alpha_{ik}, \alpha_{jk}] &= 1, & \text{for } i, j, k \text{ distinct.} \end{aligned}$$

In particular, $wP_1 = \{1\}$ and $wP_2 = F_2$.

Consider now the upper welded pure braid group (or, the upper McCool group) $wP_n^+ = wP_n \cap \text{IA}_n^+$. This is the subgroup of wP_n generated by all the automorphisms α_{ij} with $i < j$. It readily seen that $wP_1^+ = \{1\}$, $wP_2^+ = \mathbb{Z}$, and $wP_3^+ \cong F_2 \times \mathbb{Z}$. Furthermore, as shown by F. Cohen et al. [28], the group wP_n^+ can be decomposed as an iterated semi-direct product of free groups,

$$wP_n^+ = F_{n-1} \rtimes_{\alpha_{n-1}^+} wP_{n-1}^+ = F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_1, \quad (5)$$

where $\alpha_{n-1}^+ : wP_{n-1}^+ \hookrightarrow \text{Aut}(F_{n-1})$ is the restriction of the Artin representation of B_{n-1} to wP_{n-1}^+ .

It follows from the previous discussion that $P_n \cong wP_n^+$ for $n \leq 3$. In view of this fact, a natural question (asked by F. Cohen et al. in [28]) is whether the groups P_n and wP_n^+ are isomorphic for $n \geq 4$. A negative answer will be given in Corollary 17. In the same circle of ideas, let us also mention the following result from [71].

Proposition 1 ([71]). *For each $n \geq 4$, the inclusion map $wP_n^+ \hookrightarrow wP_n$ is not a split monomorphism.*

The proof of this proposition is based upon the contrasting nature of the resonance varieties of the two groups. We will come back to this point in §3.

Cohen and Pruidze showed in [22] that the center of the group wP_n^+ ($n \geq 2$) is infinite cyclic, generated by the automorphism $\prod_{1 \leq j \leq n-1} \alpha_{j,n}$. On the other hand, Dies and Nicas showed in [31] that the center of the group wP_n is trivial for $n \geq 2$.

2.3 Virtual braid groups

Closely related are the virtual braid groups νB_n , the virtual pure braid groups νP_n , and their upper triangular subgroups, νP_n^+ , obtained by omitting certain commutation relations from the respective McCool groups. The group νB_n admits a presentation with generators σ_i and s_i for $i = 1, \dots, n-1$, subject to the relations (R1), (R2), and (R3). The virtual pure braid group νP_n is defined as the kernel of the canonical epimorphism $\psi : \nu B_n \rightarrow S_n$ given by $\psi(\sigma_i) = \psi(s_i) = s_i$ for $1 \leq i \leq n-1$, see [5].

A finite presentation for νP_n was given by Bardakov [5]. The virtual pure braid group νP_n and its ‘upper’ subgroup, νP_n^+ , were both studied in depth (under different names) by Bartholdi et al. and Lee in [11, 50]. These groups are generated by

elements x_{ij} for $i \neq j$ (respectively, for $i < j$), subject to the relations

$$\begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij}, & \text{for } i, j, k \text{ distinct,} \\ [x_{ij}, x_{st}] &= 1, & \text{if } \{i, j\} \cap \{s, t\} = \emptyset. \end{aligned}$$

Unlike the inclusion map $wP_n^+ \hookrightarrow wP_n$ from Proposition 1, the inclusion $vP_n^+ \hookrightarrow vP_n$ does admit a splitting, see [11, 70].

Proposition 2. *There exist monomorphisms and epimorphisms making the following diagram commute.*

$$\begin{array}{ccccc} B_n & \xleftarrow{\varphi_n} & vB_n & \xrightarrow{\pi_n} & wB_n \\ \uparrow & & \uparrow & & \uparrow \\ P_n & \xleftarrow{\quad} & vP_n & \xrightarrow{\quad} & wP_n \end{array}$$

Furthermore, the compositions of the horizontal homomorphisms are also injective.

Proof. There are natural inclusions $\varphi_n: B_n \hookrightarrow vB_n$ and $\psi_n: B_n \hookrightarrow wB_n$ that send σ_i to σ_i , as well as a canonical projection $\pi_n: vB_n \twoheadrightarrow wB_n$, that matches the generators σ_i and s_i of the respective groups. By construction, we have that $\pi_n \circ \varphi_n = \psi_n$.

We claim that these homomorphisms restrict to homomorphisms between the respective pure-braid like groups. Indeed, as shown Bartholdi et al. in [11], the homomorphism φ_n restricts to a map $P_n \hookrightarrow vP_n$, given by

$$A_{ij} \mapsto x_{j-1,j} \dots x_{i+1,j} x_{i,j} x_{j,i} (x_{j-1,j} \dots x_{i+1,j})^{-1}. \quad (6)$$

Clearly, the projection π_n restricts to a map $vP_n \twoheadrightarrow wP_n$ that sends x_{ij} to α_{ij} . Using these observations, together with work of Bardakov [5], we see that the homomorphism ψ_n restricts to an injective map $P_n \hookrightarrow wP_n$. \square

From the defining presentations, it is readily seen that $vP_2^+ \cong \mathbb{Z}$ and $vP_2 \cong F_2$, while $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Moreover, using a computation of Bardakov et al. [10], we show in [70] that $vP_3 \cong \bar{P}_4 * \mathbb{Z}$. Consequently vP_2 , vP_3 and vP_3^+ have trivial centers.

More generally, Dies and Nicas showed in [31] that the center of the group vP_n is trivial for $n \geq 2$; furthermore, the center of vP_n^+ is trivial for $n \geq 3$, with one possible exception (and no exception if Wilf's conjecture is true).

Remark 3. In Lemma 6 from [5], Bardakov states that the group vP_n splits as a semi-direct product of the form $V_n^* \rtimes vP_{n-1}$, where V_n^* is a free group. This lemma would imply, via an easy induction argument, that $Z(vP_n) = \{1\}$ for $n \geq 2$ and $Z(vP_n^+) = \{1\}$ for $n \geq 3$. Unfortunately, there seems to be a problem with this lemma, according to the penultimate remark from [39, §6].

2.4 Topological interpretations

All the braid-like groups mentioned previously admit nice topological interpretations. For instance, the braid group B_n can be realized as the mapping class group of the 2-disk with n marked points, $\text{Mod}_{0,n}^1$, while the pure braid group P_n can be viewed as the fundamental group of the configuration space of n ordered points on the complex line, $\text{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$, see for instance [17].

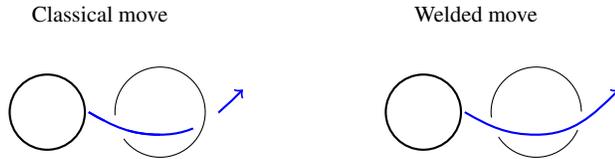


Fig. 3 Untwisted flying rings.

The welded braid group wB_n is the fundamental group of the ‘untwisted ring space,’ which consists of all configurations of n parallel rings (i.e., unknotted circles) in \mathbb{R}^3 , see Figure 3. However, as shown in [18], this space is not aspherical. The welded pure braid group wP_n can be viewed as the pure motion group of n unknotted, unlinked circles in \mathbb{R}^3 , cf. Goldsmith [40]. The group wP_n^+ is the fundamental group of the subspace consisting of all configurations of circles of unequal diameters in the ‘untwisted ring space,’ see Brendle–Hatcher [18] and Bellingeri–Bodin [13].

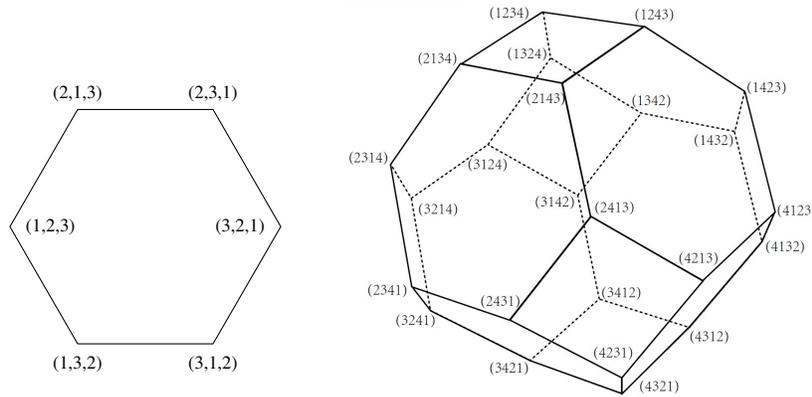


Fig. 4 P_3 and P_4 .

A classifying space for the group vP_n^+ is identified in [11] as the quotient space of the $(n - 1)$ -dimensional permutahedron P_n by actions of certain symmetric groups. More precisely, let P_n be the convex hull of the orbit of a generic point in \mathbb{R}^n under

the permutation action of the symmetric group S_n on its coordinates. Then \mathbf{P}_n is a polytope whose faces are indexed by all ordered partitions of the set $[n] = \{1, \dots, n\}$; see Figure 4. For each $r \in [n]$, there is a natural action of S_r on the disjoint union of all $(n-r)$ -dimensional faces of this polytope, $C_1 \sqcup \dots \sqcup C_r$. Similarly, a classifying space for vP_n can be constructed as a quotient space of $\mathbf{P}_n \times S_n$.

3 Cohomology algebras and resonance varieties

3.1 Cohomology algebras

Recall that the pure braid group P_n is the fundamental group of the configuration space $\text{Conf}_n(\mathbb{C})$. As shown by Arnol'd in [2], the cohomology ring $H^*(P_n; \mathbb{Z})$ is the skew-commutative ring generated by degree 1 elements u_{ij} ($1 \leq i < j \leq n$), identified with the de Rham cocycles $\frac{dz_i - dz_j}{z_i - z_j}$. The cohomology algebras of the other pure braid-like groups were computed by several groups of researchers over the last ten years, see [44, 28, 11, 50].

We summarize these computations, as follows. To start with, we denote the standard (degree 1) generators of $H^*(wP_n; \mathbb{C})$ and $H^*(vP_n; \mathbb{C})$ by a_{ij} for $1 \leq i \neq j \leq n$, and we denote the generators of $H^*(wP_n^+; \mathbb{C})$ and $H^*(vP_n^+; \mathbb{C})$ by e_{ij} for $1 \leq i < j \leq n$. Next, we list several types of relations that occur in these algebras.

$$u_{jk}u_{ik} = u_{ij}(u_{ik} - u_{jk}) \quad \text{for } i < j < k, \quad (11)$$

$$a_{ij}a_{ji} = 0 \quad \text{for } i \neq j, \quad (12)$$

$$a_{kj}a_{ik} = a_{ij}(a_{ik} - a_{jk}) \quad \text{for } i, j, k \text{ distinct}, \quad (13)$$

$$a_{ji}a_{ik} = (a_{ij} - a_{ik})a_{jk} \quad \text{for } i, j, k \text{ distinct}, \quad (14)$$

$$e_{ij}(e_{ik} - e_{jk}) = 0 \quad \text{for } i < j < k, \quad (15)$$

$$(e_{ij} - e_{ik})e_{jk} = 0 \quad \text{for } i < j < k. \quad (16)$$

Finally, we record in Table 3 the cohomology algebras of the pure braid groups, the welded pure braid groups, the virtual pure braid groups, and their upper triangular subgroups.

Table 3 Cohomology algebras of the pure braid-like groups.

	$H^*(P_n; \mathbb{C})$ [2]	$H^*(wP_n; \mathbb{C})$ [44]	$H^*(wP_n^+; \mathbb{C})$ [28]	$H^*(vP_n; \mathbb{C})$ [11, 50]	$H^*(vP_n^+; \mathbb{C})$ [11, 50]
Generators	u_{ij} $1 \leq i < j \leq n$	a_{ij} $1 \leq i \neq j \leq n$	e_{ij} $1 \leq i < j \leq n$	a_{ij} $1 \leq i \neq j \leq n$	e_{ij} $1 \leq i < j \leq n$
Relations	(11)	(12) (13)	(15)	(12) (13) (14)	(15) (16)

The above presentations of the cohomology algebras differ slightly from those given in the original papers. Using these presentations, it is easily seen that the aforementioned homomorphisms, $f_n: vP_n \hookrightarrow wP_n$ and $g_n: vP_n^+ \hookrightarrow wP_n^+$, induce epimorphisms in cohomology with \mathbb{C} -coefficients.

Let us also highlight the fact that the cohomology algebras of the pure braid-like groups are *quadratic algebras*. More precisely, they are all algebras of the form $A = E/I$, where E is an exterior algebra generated in degree 1, and I is an ideal generated in degree 2.

It is also known that the cohomology algebras of all these groups (with the exception of wP_n for $n \geq 4$) are *Koszul algebras*. That is to say, for each such algebra A , the ground field \mathbb{C} admits a free, *linear* resolution over A , or equivalently, $\text{Tor}_i^A(\mathbb{C}, \mathbb{C})_j = 0$ for $i \neq j$. In fact, it is known that in all these cases, the relations specified in Table 3 form a quadratic Grbner basis for the ideal of relations I , a fact which implies Koszulness for the algebra $A = E/I$. On the other hand, it was recently shown by Conner and Goetz [29] that the cohomology algebras of the groups wP_n are not Koszul for $n \geq 4$.

For a summary of the above discussion, as well as detailed references for the various statements therein we refer to Table 1.

3.2 Resonance varieties

Now let $A = \bigoplus_{i \geq 0} A^i$ be a graded, graded-commutative algebra over \mathbb{C} . We shall assume that A is connected, i.e., $A^0 = \mathbb{C}$, generated by the unit 1. For each element $a \in A^1$, one can form a cochain complex, (A, δ_a) , known as the *Aomoto complex*, with differentials $\delta_a^i: A^i \rightarrow A^{i+1}$ given by $\delta_a^i(u) = a \cdot u$.

The *resonance varieties* of the graded algebra A are the jump loci for the cohomology of the Aomoto complexes parametrized by the vector space A^1 . More precisely, for each $i \geq 0$ and $s \geq 1$, the (degree i , depth s) resonance variety of A is the set

$$\mathcal{R}_s^i(A) = \{a \in A^1 \mid \dim H^i(A, \delta_a) \geq s\}. \quad (7)$$

If A is locally finite (i.e., each graded piece A^i is finite-dimensional), then all these sets are closed, homogeneous subvarieties of the affine space A^1 .

The resonance varieties of a finitely-generated group G are defined as $\mathcal{R}_s^i(G) := \mathcal{R}_s^i(A)$, where $A = H^*(G, \mathbb{C})$ is the cohomology algebra of the group. If G admits a classifying space with finite k -skeleton, for some $k \geq 1$, then the sets $\mathcal{R}_s^i(G)$ are algebraic subvarieties of the affine space A^1 , for all $i \leq k$, see [60, Corollary 4.2]. We will focus here on the first resonance variety $\mathcal{R}_1(G) := \mathcal{R}_1^1(G)$, which can be described more succinctly as

$$\mathcal{R}_1(G) = \{a \in A^1 \mid \exists b \in A^1 \setminus \mathbb{C} \cdot a \text{ such that } ab = 0 \in A^2\}. \quad (8)$$

The idea of studying a family of cochain complexes parametrized by the cohomology ring in degree 1 originated from the theory of hyperplane arrangements, in

the work of Falk [34], while the more general concept from (8) first appeared in work of Matei and Suciu [55].

The resonance varieties of a variety spaces and groups have been studied intensively from many perspectives and in varying degrees of generality, see for instance [32, 68, 60] and reference therein. In particular, the first resonance varieties of the groups P_n , wP_n , and wP_n^+ have been completely determined in [27, 21, 71]. We summarize those results as follows.

Theorem 4 ([27]). *For each $n \geq 3$, the first resonance variety of the pure braid group P_n has decomposition into irreducible components given by*

$$\mathcal{R}_1(P_n) = \bigcup_{1 \leq i < j < k \leq n} L_{ijk} \cup \bigcup_{1 \leq i < j < k < l \leq n} L_{ijkl},$$

where

- L_{ijk} is the plane defined by the equations $\sum_{\{p,q\} \subset \{i,j,k\}} x_{pq} = 0$ and $x_{st} = 0$ if $\{s,t\} \not\subset \{i,j,k\}$;
- L_{ijkl} is the plane defined by the equations $\sum_{\{p,q\} \subset \{i,j,k,l\}} x_{pq} = 0$, $x_{ij} = x_{kl}$, $x_{jk} = x_{il}$, $x_{ik} = x_{jl}$ and $x_{st} = 0$ if $\{s,t\} \not\subset \{i,j,k,l\}$.

Theorem 5 ([21]). *For each $n \geq 2$, the first resonance variety of the group wP_n has decomposition into irreducible components given by*

$$\mathcal{R}_1(wP_n) = \bigcup_{1 \leq i < j \leq n} L_{ij} \cup \bigcup_{1 \leq i < j < k \leq n} L_{ijk},$$

where L_{ij} is the plane defined by the equations $x_{pq} = 0$ if $\{p,q\} \neq \{i,j\}$ and L_{ijk} is the 3-dimensional linear subspace defined by the equations $x_{ji} + x_{ki} = x_{ij} + x_{kj} = x_{ik} + x_{jk} = 0$ and $x_{st} = 0$ if $\{s,t\} \not\subset \{i,j,k\}$.

Theorem 6 ([71]). *For each $n \geq 2$, the first resonance variety of the group wP_n^+ has decomposition into irreducible components given by*

$$\mathcal{R}_1(wP_n^+) = \bigcup_{n-1 \geq i > j \geq 1} L_{i,j},$$

where $L_{i,j}$ is the linear subspace of dimension $j+1$ spanned by $e_{i+1,j+1}$ and $e_{j+1,k} - e_{i+1,k}$, for $1 \leq k \leq j$.

Much less is known about the resonance varieties of the virtual pure braid groups. For low values of n , the varieties $\mathcal{R}_1(vP_n)$ and $\mathcal{R}_1(vP_n^+)$ have been determined in [70]. For instance, the decomposition $vP_3 \cong \bar{P}_4 * \mathbb{Z}$ and known properties of resonance varieties of free products leads to the equality $\mathcal{R}_1(vP_3) = H^1(vP_3, \mathbb{C})$. We will need the following computation later on.

Proposition 7 ([70]). *The resonance variety $\mathcal{R}_1(vP_4^+)$ is the degree 6, irreducible, 3-dimensional subvariety of the affine space $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ defined by the equations*

$$\begin{aligned}
x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) &= 0, \\
x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) &= 0, \\
x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0, \\
x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) &= 0.
\end{aligned}$$

4 Lie algebras and formality

4.1 The associated graded Lie algebra of a group

The *lower central series* of a group G is a descending sequence of normal subgroups, $\{\gamma_k G\}_{k \geq 1}$, defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$. The successive quotients of this series, $\gamma_k G / \gamma_{k+1} G$, are abelian groups. The direct sum of these groups,

$$\mathrm{gr}(G; \mathbb{Z}) = \bigoplus_{k \geq 1} \gamma_k G / \gamma_{k+1} G, \quad (9)$$

endowed with the Lie bracket $[x, y]$ induced from the group commutator, has the structure of a graded Lie algebra over \mathbb{Z} . The *associated graded Lie algebra* of G over \mathbb{C} is defined as $\mathrm{gr}(G) = \mathrm{gr}(G; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Before proceeding, let us recall the following lemma due to Falk and Randell [36].

Lemma 8 ([36]). *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a split exact sequence of groups, and suppose Q acts trivially on $N/[N, N]$. Then, for each $k \geq 1$, there is an induced split exact sequence, $1 \rightarrow \gamma_k(N) \rightarrow \gamma_k(G) \rightarrow \gamma_k(Q) \rightarrow 1$.*

If a group G is finitely generated, its lower central series quotients are finitely generated abelian groups. Set

$$\phi_k(G) = \mathrm{rank} \gamma_k G / \gamma_{k+1} G, \quad (10)$$

or, equivalently, $\phi_k(G) = \dim \mathrm{gr}_k(G)$. These LCS ranks can be computed from the Hilbert series of the universal enveloping algebra of $\mathrm{gr}(G)$ by means of the Poincaré–Birkhoff–Witt theorem, as follows:

$$\prod_{k \geq 1} (1 - t^k)^{-\phi_k(G)} = \mathrm{Hilb}(U(\mathrm{gr}(G)), t). \quad (11)$$

A finitely generated group G is said to be *graded-formal* if the associated graded Lie algebra $\mathrm{gr}(G)$ is quadratic, that is, admits a presentation with generators in degree 1 and relators in degree 2. The next theorem was proved under some stronger hypothesis by Papadima and Yuzvinsky in [61], and essentially in this form in [69]. For completeness, we sketch the proof.

Theorem 9 ([61, 69]). *If G is a finitely generated, graded-formal group, and its cohomology algebra, $A = H^*(G; \mathbb{C})$, is Koszul, then the LCS ranks of G are given*

by

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = \text{Hilb}(A, -t). \quad (12)$$

Proof. By assumption, the graded Lie algebra $\mathfrak{g} = \text{gr}(G)$ is quadratic, while the algebra $A = H^*(G; \mathbb{C})$ is Koszul, hence quadratic. Write $A = T(V)/I$, where $T(V)$ is the tensor algebra on a finite-dimensional \mathbb{C} -vector space V concentrated in degree 1, and I is an ideal generated in degree 2. Define the quadratic dual of A as $A^\perp = T(V^*)/I^\perp$, where V^* is the vector space dual to V and I^\perp is the ideal generated by all $\alpha \in V^* \wedge V^*$ with $\alpha(I_2) = 0$, see [64]. It follows from [61, Lemma 4.1] that A^\perp is isomorphic to $U(\mathfrak{g})$.

Now, since A is Koszul, its quadratic dual is also Koszul. Thus, the following Koszul duality formula holds:

$$\text{Hilb}(A, t) \cdot \text{Hilb}(A^\perp, -t) = 1. \quad (13)$$

Putting things together and using (11) completes the proof. \square

4.2 The LCS ranks of the pure-braid like groups

We now turn to the associated graded Lie algebras of the pure braid-like groups. We start by listing the types of relations occurring in these Lie algebras:

$$[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0 \quad \text{for distinct } i, j, k, \quad (\text{L1})$$

$$[x_{ij}, x_{kl}] = 0 \quad \text{for } \{i, j\} \cap \{k, l\} = \emptyset, \quad (\text{L2})$$

$$[x_{ik}, x_{jk}] = 0 \quad \text{for distinct } i, j, k, \quad (\text{L3})$$

$$[x_{im}, x_{ij} + x_{ik} + x_{jk}] = 0 \quad \text{for } m = j, k \text{ and } i, j, m \text{ distinct.} \quad (\text{L4})$$

The corresponding presentations for the associated graded Lie algebras of the groups P_n , wP_n , wP_n^+ , vP_n , and vP_n^+ are summarized in Table 4. It is readily seen that all these graded Lie algebras are quadratic. Consequently, all aforementioned pure braid-like groups are graded-formal.

Table 4 Associated graded Lie algebras of the pure braid-like groups.

	$\text{gr}(P_n)$ [48, 35]	$\text{gr}(wP_n)$ [44]	$\text{gr}(wP_n^+)$ [28]	$\text{gr}(vP_n)$ [11, 50]	$\text{gr}(vP_n^+)$ [11, 50]
Generators	$x_{ij},$ $1 \leq i < j \leq n$	$x_{ij},$ $1 \leq i \neq j \leq n$	$x_{ij},$ $1 \leq i < j \leq n$	$x_{ij},$ $1 \leq i \neq j \leq n$	$x_{ij},$ $1 \leq i < j \leq n$
Relations	(L2) (L4)	(L1) (L2) (L3)	(L1) (L2) (L3)	(L1) (L2)	(L1) (L2)

The various homomorphisms between the pure-braid like groups defined previously induce morphisms between the corresponding associated graded Lie algebras. These morphisms fit into the following commuting diagram.

$$\begin{array}{ccc}
 \mathrm{gr}(vP_n^+) & \hookrightarrow & \mathrm{gr}(vP_n) \\
 \downarrow & & \downarrow \\
 \mathrm{gr}(wP_n^+) & \hookrightarrow & \mathrm{gr}(wP_n)
 \end{array}
 \begin{array}{c}
 \swarrow \\
 \mathrm{gr}(P_n) \\
 \searrow
 \end{array}
 \quad (14)$$

As shown by Bartholdi et al. in [11], the morphism $\mathrm{gr}(\varphi_n): \mathrm{gr}(P_n) \rightarrow \mathrm{gr}(vP_n)$ is injective. Using the presentations given in Table 4, we see that the remaining morphisms are either injective or surjective (as indicated in the diagram), with the possible exception of $\mathrm{gr}(\psi_n): \mathrm{gr}(P_n) \rightarrow \mathrm{gr}(wP_n)$, whose injectivity has not been established, as far as we know.

The LCS ranks of the pure braid groups P_n were computed by Kohno [49], using methods from rational homotopy theory, and by Falk and Randell [35], using the decomposition (3) and Lemma 8. The LCS ranks of the upper welded braid groups wP_n^+ were computed by F. Cohen et al. [28], using the decomposition (5) and again Lemma 8. Finally, work of Bartholdi et al. [11] and P. Lee [50] gives the LCS ranks of vP_n and vP_n^+ . We summarize these results in the next theorem.

Theorem 10. *The LCS ranks of the groups $G = P_n, wP_n^+, vP_n,$ and vP_n^+ are given by the identity $\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = \mathrm{Hilb}(H^*(G; \mathbb{C}), -t)$, with the relevant Hilbert series given in Table 1.*

Alternatively, this result follows from Theorem 9 once it is shown that, in all these cases, the Lie algebra $\mathrm{gr}(G)$ is quadratic and the cohomology algebra $A = H^*(G; \mathbb{C})$ is Koszul.

On the other hand, as mentioned previously, the cohomology algebras $H^*(wP_n, \mathbb{C})$ are not Koszul for $n \geq 4$. The (computer-aided) proof of this fact by Conner and Goetz [29] implies that the LCS ranks of wP_n are *not* given by formula (12). For instance, the formula would say that the first eight LCS ranks of wP_4 are 12, 18, 60, 180, 612, 2010, 7020, and 24480. Using the computations from [29], we see that the first seven values of $\phi_k(wP_4)$ are correct, but that $\phi_8(wP_4) = 24490$.

4.3 Residual properties

Let \mathcal{P} be a group-theoretic property. A group G is said to be *residually* \mathcal{P} if for any $g \in G, g \neq 1$, there exists a group Q satisfying property \mathcal{P} , and an epimorphism $\psi: G \rightarrow Q$ such that $\psi(g) \neq 1$.

We are mainly interested here in the residual properties related to the lower central series of G . For instance, we say that the group G is *residually nilpotent* if every non-trivial element can be detected in a nilpotent quotient. This happens precisely when the nilpotent radical of G is trivial, that is, $\bigcap_{k \geq 1} \gamma_k G = \{1\}$.

Likewise, we say that a group G is *residually torsion-free nilpotent* if every non-trivial element can be detected in a torsion-free nilpotent quotient. This happens precisely when $\bigcap_{k \geq 1} \tau_k G = \{1\}$, where

$$\tau_k G = \{g \in G \mid g^n \in \gamma_k G, \text{ for some } n \in \mathbb{N}\}. \quad (15)$$

Clearly, the second property is stronger than the first. Nevertheless, the following holds: if G is residually nilpotent and $\text{gr}_k(G, \mathbb{Z})$ is torsion-free, for each $k \geq 1$, then G is residually torsion-free nilpotent. In turn, this last property implies that G is torsion-free. Moreover, residually nilpotent groups are residually finite.

For a group G , the properties of being residually nilpotent or residually torsion-free nilpotent are inherited by subgroups $H < G$, since $\gamma_k H < \gamma_k G$ and $\tau_k H < \tau_k G$. Moreover, both properties are preserved under direct products and free products, see Malcev [52] and Baumslag [12]. For more on this subject, see also [6, 7, 46].

In [1], Andreadakis introduced a new filtration on the automorphism group of a group G , nowadays called the Andreadakis–Johnson filtration. This filtration is defined by setting

$$\Phi_k(\text{Aut}(G)) = \ker(\text{Aut}(G) \rightarrow \text{Aut}(G/\gamma_{k+1}(G))), \quad (16)$$

for all $k \geq 0$. Note that $\Phi_0(\text{Aut}(G)) = \text{Aut}(G)$; the group $\mathcal{I}(G) = \Phi_1(\text{Aut}(G))$ is called the *Torelli group* of G .

As shown by Andreadakis, if the intersection $\bigcap_{k \geq 1} \gamma_k G$ is trivial then the intersection $\bigcap_{k \geq 1} \Phi_k(\text{Aut}(G))$ is also trivial. Furthermore, a theorem of L. Kaloujnine implies that $\gamma_k(\mathcal{I}(G)) < \Phi_k(\text{Aut}(G))$ for all $k \geq 1$, see e.g. [59]. We thus have the following basic result.

Theorem 11 ([1]). *Let G be a residually nilpotent group. Then the Torelli group $\mathcal{I}(G)$ is also residually nilpotent.*

As noted by Hain [43] in the case of the Torelli group of a Riemann surface and proved by Berceanu and Papadima [15] in full generality, a stronger assumption leads to a stronger conclusion.

Theorem 12 ([43, 15]). *Let G be a finitely generated, residually nilpotent group, and suppose $\text{gr}_k(G, \mathbb{Z})$ is torsion-free for all $k \geq 1$. Then the Torelli group $\mathcal{I}(G)$ is residually torsion-free nilpotent.*

We specialize now to the case $G = F_n$. In [51] Magnus showed that all free groups are residually torsion-free nilpotent (this also follows from the aforementioned results of Malcev and Baumslag). Furthermore, P. Hall and Magnus showed that $\text{gr}(F_n, \mathbb{Z})$ is the free Lie algebra on n generators, and thus is torsion-free (see [66, Ch. IV, §6]). Therefore, by Theorem 12, the Torelli group $\text{IA}_n = \mathcal{I}(F_n)$ is residually torsion-free nilpotent. Hence, all its subgroups, such as IA_n^+ , P_n , wP_n , and wP_n^+ also enjoy this property.

Let us now look in more detail at the residual properties of the braid groups and their relatives. We start with the classical braid groups. As shown by Kramer [47]

and Bigelow [16], the braid groups B_n admit faithful linear representations, and thus, by a theorem of Malcev, they are residually finite. On the other hand, it was shown in [41] by Gorin and Lin that $\gamma_2 B_n = \gamma_3 B_n$ for $n \geq 3$ (see [14] for an alternative proof); thus, the braid groups B_n are not residually nilpotent for $n \geq 3$. Since both the welded braid group wB_n and the virtual braid group vB_n contain the braid group B_n as a subgroup, we conclude that wB_n and vB_n are not residually nilpotent for $n \geq 3$, either (see also [6]).

A different approach to the residual properties of the pure braid groups was taken by Falk and Randell in [36]. Using the semi-direct product decomposition (3) and Lemma 8, these authors gave another proof that the groups P_n are residually nilpotent; in fact, their proof shows that P_n is residually torsion-free nilpotent, see [6]. A similar approach, based on decomposition (5) provides another proof that the upper McCool groups wP_n^+ are residually torsion-free nilpotent.

From the work of Berceanu and Papadima [15] mentioned above, we know that the full McCool groups wP_n are residually torsion-free nilpotent. For $n = 3$, an even stronger result was proved by Metaftsis and Papistas [57], who showed that $\text{gr}_k(wP_3, \mathbb{Z})$ is torsion-free for all k . Whether an analogous statement holds for the McCool groups wP_n with $n \geq 4$ is an open problem.

Finally, let us consider the virtual pure braid groups. Clearly, the groups $vP_2^+ = \mathbb{Z}$, $vP_2 = F_2$, and $vP_3^+ = \mathbb{Z} * \mathbb{Z}^2$ are all residually torsion-free nilpotent. Bardakov et al. [10] show that the group vP_3 also enjoys this property. Alternatively, this can be seen from our decomposition $vP_3 \cong \bar{P}_4 * \mathbb{Z}$, the fact that both \bar{P}_4 and \mathbb{Z} have this property, and the aforementioned result of Malcev. Whether or not the groups vP_n and vP_n^+ are residually torsion-free nilpotent for $n \geq 4$ is an open problem, see [6].

4.4 Malcev Lie algebras and formality properties

For each finitely-generated, torsion-free nilpotent group N , A.I. Malcev [53] constructed a filtered Lie algebra $\mathfrak{m}(N)$ over \mathbb{Q} , which is now called the Malcev Lie algebra of N . Given a finitely generated group G , the inverse limit of the Malcev Lie algebras of the torsion free parts of the nilpotent quotients $G/\gamma_i G$ for $i \geq 2$ defines the *Malcev Lie algebra* of the group G , which we denote by $\mathfrak{m}(G)$. This Lie algebra coincides with the dual Lie algebra of the 1-minimal model of G defined by D. Sullivan. For relevant background, we refer to [69] and references therein.

We will use another equivalent definition of the Malcev Lie algebra which was given by Quillen in [62]. The group-algebra $\mathbb{C}G$ has a natural Hopf algebra structure with comultiplication $\Delta : \mathbb{C}G \otimes \mathbb{C}G \rightarrow \mathbb{C}G$ given by $\Delta(x) = x \otimes x$ for $x \in G$. Let $\widehat{\mathbb{C}G} = \varprojlim_r \mathbb{C}G/I^r$ be the completion of $\mathbb{C}G$ with respect to the I -adic filtration, where I is the augmentation ideal. An element $x \in \widehat{\mathbb{C}G}$ is called *primitive* if $\widehat{\Delta}x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The Malcev Lie algebra $\mathfrak{m}(G)$ is then the set of all primitive elements in $\widehat{\mathbb{C}G}$, with Lie bracket $[x, y] = xy - yx$, and endowed with the induced filtration.

The group G is said to be *filtered-formal* if there exists an isomorphism of filtered Lie algebras, $\mathfrak{m}(G) \cong \widehat{\text{gr}}(G)$. The group G is *1-formal* if there exists a filtered Lie algebra isomorphism $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}$, where \mathfrak{h} is a quadratic Lie algebra. As shown in [69], a finitely generated group G is 1-formal if and only if it is both graded-formal and filtered-formal.

Theorem 13. *For each $n \geq 1$, the following hold.*

1. ([2, 48]) *The pure braid group P_n is 1-formal.*
2. ([15]) *The welded pure braid groups wP_n and wP_n^+ are 1-formal.*
3. ([11, 50]) *The virtual pure braid groups vP_n and vP_n^+ are graded-formal.*

Let us now recall the following consequence of the ‘Tangent Cone Theorem’ from [32].

Theorem 14 ([32]). *Let G be a finitely generated, 1-formal group. Then all irreducible components of $\mathcal{R}_1(G)$ are rationally defined linear subspaces of $H^1(G, \mathbb{C})$.*

In view of Proposition 7 and Theorem 14, the group vP_4^+ is not 1-formal. In fact, we have the following theorem.

Theorem 15 ([71]). *For the virtual pure braid group vP_n and its upper-triangular subgroup, vP_n^+ , the following hold: both are 1-formal for $n \leq 3$, and both are non-filtered-formal for $n \geq 4$.*

4.5 Chen ranks

The *Chen Lie algebra* of a finitely generated group G is defined to be the associated graded Lie algebra of its second derived quotient, G/G'' . The projection $\pi: G \rightarrow G/G''$ induces an epimorphism, $\text{gr}(\pi): \text{gr}(G) \rightarrow \text{gr}(G/G'')$. It is readily verified that $\text{gr}_k(\pi)$ is an isomorphism for $k \leq 3$.

In [19], K.-T. Chen gave a method for finding a basis for $\text{gr}(G/G'')$ via a path integral technique for free groups. In the process, he showed that the Chen ranks of the free group of rank n are given by $\theta_1(F_n) = n$ and

$$\theta_k(F_n) = \binom{n+k-2}{k} (k-1), \quad \text{for } k \geq 2. \quad (17)$$

As shown by Massey in [54], the Chen ranks of a group G can be computed from the Alexander invariant G'/G'' . In [24, 26], Cohen and Suciú developed this method, by introducing the use of Gröbner basis techniques in this context. As an application, they showed in [24] that the Chen ranks of the pure braid groups P_n are given by

$$\theta_k(P_n) = (k-1) \binom{n+1}{4}, \quad \text{for } k \geq 3. \quad (18)$$

Based on this and many other similar computations, the first author conjectured in [67] that for $k \gg 0$, the Chen ranks of an arrangement group G are given by

$$\theta_k(G) = \sum_{m \geq 2} c_m \cdot \theta_k(F_m), \quad (19)$$

where c_m is the number of m -dimensional components of $\mathcal{R}_1(G)$. Much work has gone into proving this conjecture, with special cases being verified in [63, 58, 65]. A key advance was made in [58], where it was shown that the Chen ranks of a finitely presented, 1-formal group G are determined by the truncated cohomology ring $H^{\leq 2}(G, \mathbb{C})$.

Using this fact, Cohen and Schenck show in [23] that, for a finitely presented, commutator-relators 1-formal group G , the Chen ranks formula (19) holds, provided the components of $\mathcal{R}_1(G)$ are isotropic, projectively disjoint, and reduced as a scheme. They also verify that arrangement groups and the welded pure braid groups wP_n satisfy these conditions. From the Chen ranks formula (19) and the first resonance varieties of wP_n in (5), they deduce that for $k \gg 1$, the Chen ranks of wP_n are given by

$$\theta_k(wP_n) = (k-1) \binom{n}{2} + (k^2-1) \binom{n}{3}. \quad (20)$$

We conjecture that formula (20) holds for all n and all $k \geq 4$. We have verified this conjecture for $n \leq 8$, based on direct computations of the Chen ranks of the groups wP_n in that range.

However, the resonance varieties of wP_n^+ do not satisfy the isotropicity hypothesis. Nevertheless, we compute in [71] the Chen ranks of these groups, using the Gröbner basis algorithm outlined in [24]. The result reads as follows.

Theorem 16 ([71]). *The Chen ranks of wP_n^+ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, $\theta_3 = 2 \binom{n+1}{4}$, and*

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^k \binom{n+i-2}{i+1} + \binom{n+1}{4}$$

for $k \geq 4$.

We have seen previously that the pure braid group P_n , the upper McCool group wP_n^+ , and the group $\Pi_n = \prod_{i=1}^{n-1} F_i$ share the same LCS ranks and the same Betti numbers. Furthermore, the centers of all these groups are infinite cyclic, provided $n \geq 2$. However, the Chen ranks can distinguish these groups.

Corollary 17 ([71]). *For $n \geq 4$, the pure braid group P_n , the upper McCool group wP_n^+ , and the group Π_n are all pairwise non-isomorphic.*

Remark 18. The fact that $P_n \not\cong wP_n^+$ for $n \geq 4$ answers in the negative Problem 1 from [28, §10]. An alternate solution for $n = 4$ was given by Bardakov and Mikhailov in [9], but that solution relies on the claim that the single-variable Alexander polynomial of a finitely presented group G is an invariant of the group, a claim which is far from being true if $b_1(G) > 1$.

The Chen ranks of the virtual pure braid groups and their upper triangular subgroups are more complicated. We summarize some of our computations of these ranks, as follows.

$$\begin{aligned}\sum_{k \geq 2} \theta_k(vP_3^+) t^{k-2} &= (2-t)/(1-t)^3, \\ \sum_{k \geq 2} \theta_k(vP_4^+) t^{k-2} &= (8-3t+t^2)/(1-t)^4, \\ \sum_{k \geq 2} \theta_k(vP_5^+) t^{k-2} &= (20+15t+5t^2)/(1-t)^4, \\ \sum_{k \geq 2} \theta_k(vP_6^+) t^{k-2} &= (40+35t-40t^2-20t^3)/(1-t)^5. \\ \sum_{k \geq 2} \theta_k(vP_3) t^{k-2} &= (9-20t+15t^2-4t^4+t^5)/(1-t)^6.\end{aligned}$$

It would be interesting to find closed formulas for the Chen ranks of the groups vP_n^+ and vP_n , but this seems to be a very challenging undertaking.

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