

FORMALITY PROPERTIES OF FINITELY GENERATED GROUPS AND LIE ALGEBRAS

ALEXANDER I. SUCIU¹ AND HE WANG

ABSTRACT. We explore the graded-formality and filtered-formality properties of finitely generated groups by studying the various Lie algebras over a field of characteristic 0 attached to such groups, including the Malcev Lie algebra, the associated graded Lie algebra, the holonomy Lie algebra, and the Chen Lie algebra. We explain how these notions behave with respect to split injections, coproducts, direct products, as well as field extensions, and how they are inherited by solvable and nilpotent quotients. A key tool in this analysis is the 1-minimal model of the group, and the way this model relates to the aforementioned Lie algebras. We illustrate our approach with examples drawn from a variety of group-theoretic and topological contexts, such as finitely generated torsion-free nilpotent groups, link groups, and fundamental groups of Seifert fibered manifolds.

1. INTRODUCTION

The main focus of this paper is on the formality properties of finitely generated groups, as reflected in the structure of the various graded or filtered Lie algebras, as well as commutative, differential graded algebras attached to such groups.

1.1. From groups to Lie algebras. Throughout, we will let G be a finitely generated group, and we will let \mathbb{k} be a coefficient field of characteristic 0. Our main focus will be on several \mathbb{k} -Lie algebras attached to such a group, and the way they all connect to each other.

By far the best known of these Lie algebras is the *associated graded Lie algebra*, $\mathrm{gr}(G; \mathbb{k})$, introduced by W. Magnus in the 1930s, and further developed by E. Witt, P. Hall, M. Lazard, and many others, cf. [51] and references therein. This is a finitely generated graded Lie algebra, whose graded pieces are the successive quotients of the lower central series of G (tensored with \mathbb{k}), and whose Lie bracket is induced from the group commutator. The quintessential example is the free Lie algebra $\mathrm{lie}(\mathbb{k}^n)$, which is the associated graded Lie algebra of the free group on n generators, F_n .

Closely related is the *holonomy Lie algebra*, $\mathfrak{h}(G; \mathbb{k})$, introduced by Kohno in [39], building on work of Chen [12], and further studied by Markl–Papadima [53] and Papadima–Suciu [59]. This is a quadratic Lie algebra, obtained as the quotient of the free Lie algebra on $H_1(G; \mathbb{k})$ by the ideal generated by the image of the dual of the cup product map in degree 1. The holonomy Lie algebra comes equipped with a natural epimorphism $\Phi_G : \mathfrak{h}(G; \mathbb{k}) \twoheadrightarrow \mathrm{gr}(G; \mathbb{k})$, and thus can be viewed as the quadratic approximation to the associated graded Lie algebra.

2010 *Mathematics Subject Classification.* Primary 20F40. Secondary 16S37, 16W70, 17B70, 20F14, 20F18, 20J05, 55P62, 57M05.

Key words and phrases. Central series, Malcev Lie algebra, holonomy Lie algebra, Chen Lie algebra, minimal model, 1-formality, graded-formality, filtered-formality, nilpotent group, Seifert manifold.

¹Supported in part by the National Science Foundation (grant DMS–1010298), the National Security Agency (grant H98230-13-1-0225), and the Simons Foundation (collaboration grant for mathematicians 354156).

The most intricate of these Lie algebras (yet, in many ways, the most important) is the *Malcev Lie algebra*, $\mathfrak{m}(G; \mathbb{k})$. As shown by Malcev in [52], every finitely generated, torsion-free nilpotent group N is the fundamental group of a nilmanifold, whose corresponding \mathbb{k} -Lie algebra is $\mathfrak{m}(N; \mathbb{k})$. Taking now the nilpotent quotients of G , we may define $\mathfrak{m}(G; \mathbb{k})$ as the inverse limit of the resulting tower of nilpotent Lie algebras, $\mathfrak{m}(G/\Gamma_k G; \mathbb{k})$. By construction, the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra; the pronilpotent group corresponding to this pronilpotent Lie algebra is denoted by $\mathfrak{M}(G; \mathbb{k})$. In [69, 70], Quillen showed that $\mathfrak{m}(G; \mathbb{k})$ is the set of all primitive elements in $\widehat{\mathbb{k}G}$, the completion of the group algebra of G with respect to the filtration by powers of the augmentation ideal, and that the associated graded Lie algebra of $\mathfrak{m}(G; \mathbb{k})$ with respect to the inverse limit filtration is isomorphic to $\mathrm{gr}(G; \mathbb{k})$. Furthermore, the set of all group-like elements in $\widehat{\mathbb{k}G}$, with multiplication and filtration inherited from $\widehat{\mathbb{k}G}$, forms a complete, filtered group isomorphic to $\mathfrak{M}(G; \mathbb{k})$.

1.2. Formality notions. In his foundational paper on rational homotopy theory [81], Sullivan associated to each path-connected space X a ‘minimal model,’ $\mathcal{M}(X)$, which can be viewed as an algebraic approximation to the space. If, moreover, X is a CW-complex with finitely many 1-cells, then the Lie algebra dual to the first stage of the minimal model is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ associated to the fundamental group $G = \pi_1(X)$. The space X is said to be *formal* if the commutative, graded differential algebra $\mathcal{M}(X)$ is quasi-isomorphic to the cohomology ring $H^*(X; \mathbb{Q})$, endowed with the zero differential. If there exists a DGA morphism from the i -minimal model $\mathcal{M}(X, i)$ to $H^*(X; \mathbb{Q})$ inducing isomorphisms in cohomology up to degree i and a monomorphism in degree $i + 1$, then X is called *i -formal*.

A finitely generated group G is said to be *1-formal* (over \mathbb{Q}) if it has a classifying space $K(G, 1)$ which is 1-formal. The study of the various Lie algebras attached to the fundamental group of a space provides a fruitful way to look at the formality problem. Indeed, the group G is 1-formal if and only if the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to the rational holonomy Lie algebra of G , completed with respect to the lower central series (LCS) filtration.

We find it useful to separate the 1-formality property of a group G into two complementary properties: graded-formality and filtered-formality. More precisely, we say that G is *graded-formal* (over \mathbb{k}) if the associated graded Lie algebra $\mathrm{gr}(G; \mathbb{k})$ is isomorphic, as a graded Lie algebra, to the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$. Likewise, we say that G is *filtered-formal* (over \mathbb{k}) if the Malcev Lie algebra $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ is isomorphic, as a filtered Lie algebra, to the completion of its associated graded Lie algebra, $\widehat{\mathrm{gr}}(\mathfrak{m})$, where both \mathfrak{m} and $\widehat{\mathrm{gr}}(\mathfrak{m})$ are endowed with the respective inverse limit filtrations. As we show in Proposition 7.6, the group G is 1-formal if and only if it is both graded-formal and filtered-formal.

All four possible combinations of these formality properties do occur. First, examples of 1-formal groups include right-angled Artin groups, groups with first Betti number equal to 0 or 1, fundamental groups of compact Kähler manifolds, and fundamental groups of complements of complex algebraic hypersurfaces. Second, there are many torsion-free, nilpotent groups (Examples 7.8 and 9.8) as well as fundamental groups of link complements (Example 7.11) which are filtered-formal, but not graded-formal. Third, there are also finitely presented groups, such as those from Examples 7.9, 7.10, and 7.12 which are graded-formal but not filtered-formal. Fourth, there are groups which enjoy none of these formality properties; indeed, if G_1 is a group of the second type and G_2 is a group of the third type, then Theorem 1.4 below shows that the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.

1.3. Field extensions and formality. We start by reviewing in §2 some basic notions pertaining to filtered and graded Lie algebras. We say that a Lie algebra \mathfrak{g} (over a field \mathbb{k} of characteristic 0) is *filtered formal* if it admits complete, separated filtration, and there exists a filtration-preserving isomorphism $\mathfrak{g} \cong \widehat{\text{gr}}(\mathfrak{g})$ which induces the identity on associated graded Lie algebras. Our first result, which generalizes a recent theorem of Cornulier [13], shows that filtered-formality behaves well with respect to field extensions. The proof we give in Theorem 2.7 is based on recent work of Enriquez [20] and Maassarani [49].

Theorem 1.1. *Let \mathfrak{g} be a \mathbb{k} -Lie algebra endowed with a complete, separated filtration such that $\text{gr}(\mathfrak{g})$ is finitely generated in degree 1. If $\mathbb{k} \subset \mathbb{K}$ is a field extension, then \mathfrak{g} is filtered-formal if and only if the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal.*

We continue in §3 with a review of the notions of quadratic and Koszul algebras. In §4, we analyze in detail the relationship between the 1-minimal model $\mathcal{M}(A, 1)$ and the dual Lie algebra $\mathfrak{L}(A)$ of a differential graded \mathbb{k} -algebra (A, d) . The reason for doing this is a result of Sullivan [81], which gives a functorial isomorphism of pronilpotent Lie algebras, $\mathfrak{L}(A) \cong \mathfrak{m}(G; \mathbb{k})$, provided $\mathcal{M}(A, 1)$ is a 1-minimal model for a finitely generated group G . The book by Félix, Halperin, and Thomas [23] provides a good reference for this subject.

Of particular interest is the case when A is a connected, graded commutative algebra with A^1 finite-dimensional, endowed with the differential $d = 0$. In Theorem 4.10, we show that $\mathfrak{L}(A)$ is isomorphic (as a filtered Lie algebra) to the degree completion of the holonomy Lie algebra of A . In the case when $A^{\leq 2} = H^{\leq 2}(G; \mathbb{k})$, for some finitely generated group G , this result recovers the aforementioned characterization of the 1-formality property of G . In Theorem 6.5 we give an alternate interpretation of filtered formality: A group G is filtered-formal if and only if G has a 1-minimal model whose differential is homogeneous with respect to the canonical Hirsch weights.

As is well-known, a space X with finite Betti numbers is formal over \mathbb{Q} if and only if it is formal over \mathbb{k} , for any field \mathbb{k} of characteristic 0. This foundational result was proved independently and in various degrees of generality by Halperin and Stasheff [33], Neisendorfer and Miller [58], and Sullivan [81]. Motivated by these classical results, as well as the aforementioned work of Cornulier, we investigate the way in which the formality properties of spaces and groups behave under field extensions. Our next result, which is a combination of Corollary 4.23, Corollary 5.10, Proposition 6.6, and Corollary 7.7, can be stated as follows.

Theorem 1.2. *Let X be a path-connected space, with finitely generated fundamental group G . Let \mathbb{k} be a field of characteristic zero, and let $\mathbb{k} \subset \mathbb{K}$ be a field extension.*

- (1) *Suppose X has finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i -formal over \mathbb{k} if and only if X is i -formal over \mathbb{K} .*
- (2) *The group G is 1-formal over \mathbb{k} if and only if G is 1-formal over \mathbb{K} .*
- (3) *The group G is graded-formal over \mathbb{k} if and only if G is graded-formal over \mathbb{K} .*
- (4) *The group G is filtered-formal over \mathbb{k} if and only if G is filtered-formal over \mathbb{K} .*

In summary, under appropriate finiteness conditions, all the formality properties that we study in this paper are independent of the ground field $\mathbb{k} \supset \mathbb{Q}$. Hence, we will sometimes avoid mentioning the coefficient field when referring to these formality notions. The descent property for partial formality from Theorem 1.2, part (1) has been used in [62] to establish the $(n - 1)$ -formality over \mathbb{Q} of compact Sasakian manifolds of dimension $2n + 1$.

1.4. Propagation of formality. Next, we turn our attention to the way in which the various formality notions for groups behave with respect to split injections, coproducts, and direct products. Our first result in this direction is a combination of Theorem 5.11 and 7.15, and can be stated as follows.

Theorem 1.3. *Let G be a finitely generated group, and let $K \leq G$ be a subgroup. Suppose there is a split monomorphism $\iota: K \rightarrow G$. Then:*

- (1) *If G is graded-formal, then K is also graded-formal.*
- (2) *If G is filtered-formal, then K is also filtered-formal.*
- (3) *If G is 1-formal, then K is also 1-formal.*

In particular, if a semi-direct product $G_1 \rtimes G_2$ has one of the above formality properties, then G_2 also has that property; in general, though, G_1 will not, as illustrated in Example 5.13.

As shown by Dimca et al. [16], both the product and the coproduct of two 1-formal groups is again 1-formal. Also, as shown by Plantiko [65], the product and coproduct of two graded-formal groups is again graded-formal. We sharpen these results in the next theorem, which is a combination of Propositions 5.15 and 7.17.

Theorem 1.4. *Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.*

- (1) *G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).*
- (2) *$G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*
- (3) *$G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*

Both Theorem 1.3 and Theorem 1.4 can be used to decide the formality properties of new groups from those of known groups. In general, though, even when both G_1 and G_2 are 1-formal, we cannot conclude that an arbitrary semi-direct product $G_1 \rtimes G_2$ is 1-formal (see Example 7.12). The various formality properties are not necessarily inherited by quotient groups. However, as we shall see in Theorem 1.5 and Theorem 7.13, respectively, filtered-formality is passed on to the derived quotients and to the nilpotent quotients of a group.

1.5. Derived series and Lie algebras. In §8, we investigate some of the relationships between the lower central series and derived series of a group and the derived series of the corresponding Lie algebras. In [11], Chen studied the lower central series quotients of the maximal metabelian quotient of a finitely generated free group, and computed their graded ranks. More generally, following Papadima and Suciu [59], we may define the *Chen Lie algebras* of a group G as the associated graded Lie algebras of its solvable quotients, $\text{gr}(G/G^{(i)}; \mathbb{k})$. Our next theorem (which combines Theorem 8.3 and Corollary 8.5) sharpens and extends the main result of [59].

Theorem 1.5. *Let G be a finitely generated group. For each $i \geq 2$, the projection $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded \mathbb{k} -Lie algebras, $\Psi_G^{(i)}: \text{gr}(G; \mathbb{k})/\text{gr}(G; \mathbb{k})^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$. Moreover,*

- (1) *If G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.*
- (2) *If G is a 1-formal group, then $\mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})^{(i)} \cong \text{gr}(G/G^{(i)}; \mathbb{k})$.*

Given a finitely presented group G , the solvable quotients $G/G^{(i)}$ need not be finitely presented (see [63]). Thus, finding presentations for the Chen Lie algebra $\text{gr}(G/G^{(i)})$ can be an arduous task. Nevertheless, Theorem 1.5 provides a method for finding such presentations, under suitable formality assumptions. The theorem can also be used as an obstruction to 1-formality.

1.6. Nilpotent groups and Lie algebras. Our techniques apply especially well to the class of finitely generated, torsion-free nilpotent groups. Carlson and Toledo [8] studied the 1-formality properties of such groups, while Plantiko [65] gave a sufficient conditions for such groups to be non-graded-formal. For nilpotent Lie algebras, the notion of filtered-formality has been studied by Leger [46], Cornulier [13], Kasuya [38], and others. In particular, it is shown in [13] that the systolic growth of a finitely generated nilpotent group G is asymptotically equivalent to its growth if and only if the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is filtered-formal (or, ‘Carnot’), while in [38] it is shown that the variety of flat connections on a filtered-formal (or, ‘naturally graded’), n -step nilpotent Lie algebra has a singularity at the origin cut out by polynomials of degree at most $n + 1$.

We investigate in §9 the filtered-formality of nilpotent groups and Lie algebras. The next result combines Theorem 9.4 and Proposition 9.9.

Theorem 1.6. *Let G be a finitely generated, torsion-free nilpotent group.*

- (1) *If G is a 2-step nilpotent group with torsion-free abelianization, then G is filtered-formal.*
- (2) *Suppose G is filtered-formal. Then the universal enveloping algebra $U(\mathrm{gr}(G; \mathbb{k}))$ is Koszul if and only if G is abelian.*

As mentioned in §1.4, nilpotent quotients of filtered-formal groups are filtered-formal; in particular, each n -step, free nilpotent group $F/\Gamma_n F$ is filtered-formal. A classical example is the unipotent group $U_n(\mathbb{Z})$, which is known to be filtered-formal by Lambe and Priddy [42], but not graded-formal for $n \geq 3$. In [13], Cornulier showed that the filtered-formality of a finite-dimensional nilpotent Lie algebra is independent of the ground field, thereby answering a question of Johnson [37]. The much more general Theorem 1.1 allows us to recover this result in Proposition 9.3.

1.7. Further applications. We end in §10 with a detailed study of fundamental groups of (orientable) Seifert fibered manifolds from a rational homotopy viewpoint. Starting from the minimal model of such a manifold M , as described in [68], we find a presentation for the Malcev Lie algebra $\mathfrak{m}(\pi_1(M); \mathbb{k})$, and we use this information to derive a presentation for $\mathrm{gr}(\pi_1(M); \mathbb{k})$. As an application, we show that Seifert manifold groups are filtered-formal, and determine precisely which ones are graded-formal. The techniques developed here have been used in [62] to prove a more general result about the filtered-formality of Sasakian groups.

This work was motivated in good part by papers [1, 7] of Etingof et al. on triangular and quasi-triangular groups, also known as the (upper) pure virtual braid groups. In [75], we apply the techniques developed here to study the formality properties of such groups. Related results for the McCool groups (or, the welded pure braid groups) and other braid-like groups are given in [76, 78]. Finally, the relationship between filtered formality and expansions of groups is explored in [79].

2. FILTERED AND GRADED LIE ALGEBRAS

In this section we study the interactions between filtered Lie algebras, their completions, and their associated graded Lie algebras, mainly as they relate to the notion of filtered-formality.

2.1. Graded Lie algebras. We start by reviewing some standard material on Lie algebras, following the exposition from [19, 66, 70, 72].

Fix a ground field \mathbb{k} of characteristic 0. Let \mathfrak{g} be a Lie algebra over \mathbb{k} , i.e., a \mathbb{k} -vector space \mathfrak{g} endowed with a bilinear operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Lie identities. We say that \mathfrak{g} is a *graded Lie algebra* if \mathfrak{g} decomposes as $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$ and the Lie bracket sends $\mathfrak{g}_i \times \mathfrak{g}_j$ to \mathfrak{g}_{i+j} , for all

i and j . A morphism of graded Lie algebras is a \mathbb{k} -linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ which preserves the Lie brackets and the degrees. In particular, φ induces \mathbb{k} -linear maps $\varphi_i: \mathfrak{g}_i \rightarrow \mathfrak{h}_i$ for all $i \geq 1$.

The most basic example of a graded Lie algebra is constructed as follows. Let V a \mathbb{k} -vector space. The tensor algebra $T(V)$ has a natural Hopf algebra structure, with comultiplication Δ and counit ε the algebra maps given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\varepsilon(v) = 0$, for $v \in V$. The *free Lie algebra* on V is the set of primitive elements, i.e., $\text{lie}(V) = \{x \in T(V) \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$, with Lie bracket $[x, y] = x \otimes y - y \otimes x$ and grading induced from $T(V)$.

A Lie algebra \mathfrak{g} is said to be *finitely generated* if there is an epimorphism $\varphi: \text{lie}(V) \rightarrow \mathfrak{g}$ for some finite-dimensional \mathbb{k} -vector space V . If, moreover, the Lie ideal $\mathfrak{r} = \ker(\varphi)$ is finitely generated as a Lie algebra, then \mathfrak{g} is called *finitely presented*. Now suppose all elements of V are assigned degree 1 in $T(V)$. Then the inclusion $\iota: \text{lie}(V) \rightarrow T(V)$ identifies $\text{lie}_1(V)$ with $T_1(V) = V$. Furthermore, ι maps $\text{lie}_2(V)$ to $T_2(V) = V \otimes V$ by sending $[v, w]$ to $v \otimes w - w \otimes v$ for each $v, w \in V$; we thus may identify $\text{lie}_2(V) \cong V \wedge V$ by sending $[v, w]$ to $v \wedge w$.

If $\mathfrak{g} = \text{lie}(V)/\mathfrak{r}$, with V a (finite-dimensional) vector space concentrated in degree 1, then we say \mathfrak{g} is (*finitely*) *generated in degree 1*. If, moreover, the Lie ideal \mathfrak{r} is homogeneous, then \mathfrak{g} is a graded Lie algebra. In particular, if \mathfrak{g} is finitely generated in degree 1 and the homogeneous ideal \mathfrak{r} is generated in degree 2, then we say \mathfrak{g} is a *quadratic Lie algebra*.

2.2. Filtrations. We will be very much interested in this work in Lie algebras endowed with a filtration, usually but not always enjoying an extra ‘multiplicative’ property. At the most basic level, a *filtration* \mathcal{F} on a Lie algebra \mathfrak{g} is a nested sequence of Lie ideals, $\mathfrak{g} = \mathcal{F}_1\mathfrak{g} \supset \mathcal{F}_2\mathfrak{g} \supset \dots$.

A well-known such filtration is the *derived series*, $\mathcal{F}_i\mathfrak{g} = \mathfrak{g}^{(i-1)}$, defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \geq 1$. The derived series is preserved by Lie algebra maps. The quotient Lie algebras $\mathfrak{g}/\mathfrak{g}^{(i)}$ are solvable; moreover, if \mathfrak{g} is a graded Lie algebra, all these solvable quotients inherit a graded Lie algebra structure.

The existence of a filtration \mathcal{F} on a Lie algebra \mathfrak{g} makes \mathfrak{g} into a topological vector space, by defining a basis of open neighborhoods of an element $x \in \mathfrak{g}$ to be $\{x + \mathcal{F}_k\mathfrak{g}\}_{k \in \mathbb{N}}$. The fact that each basis neighborhood $\mathcal{F}_k\mathfrak{g}$ is a Lie subalgebra implies that the Lie bracket map $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is continuous; thus, \mathfrak{g} is, in fact, a topological Lie algebra. We say that \mathfrak{g} is *complete* (respectively, *separated*) if the underlying topological vector space enjoys those properties.

Given an ideal $\mathfrak{a} \subset \mathfrak{g}$, there is an induced filtration on it, given by $\mathcal{F}_k\mathfrak{a} = \mathcal{F}_k\mathfrak{g} \cap \mathfrak{a}$. Likewise, the quotient Lie algebra, $\mathfrak{g}/\mathfrak{a}$, has a naturally induced filtration with terms $\mathcal{F}_k\mathfrak{g}/\mathcal{F}_k\mathfrak{a}$. Let $\overline{\mathfrak{a}}$ be the closure of \mathfrak{a} in the filtration topology. Then $\overline{\mathfrak{a}}$ is a closed ideal of \mathfrak{g} . Moreover, by the continuity of the Lie bracket, we have that

$$(1) \quad \overline{[\mathfrak{a}, \mathfrak{r}]} = [\overline{\mathfrak{a}}, \overline{\mathfrak{r}}].$$

Finally, if \mathfrak{g} is complete (or separated), then $\mathfrak{g}/\overline{\mathfrak{a}}$ is also complete (or separated).

2.3. Completions. For each $j \geq k$, there is a canonical projection $\mathfrak{g}/\mathcal{F}_j\mathfrak{g} \rightarrow \mathfrak{g}/\mathcal{F}_k\mathfrak{g}$, compatible with the projections from \mathfrak{g} to its quotient Lie algebras $\mathfrak{g}/\mathcal{F}_k\mathfrak{g}$. The *completion* of the Lie algebra \mathfrak{g} with respect to the filtration \mathcal{F} is defined as the limit of this inverse system, i.e.,

$$(2) \quad \widehat{\mathfrak{g}} := \varprojlim_k \mathfrak{g}/\mathcal{F}_k\mathfrak{g} = \{(g_1, g_2, \dots) \in \prod_{i=1}^{\infty} \mathfrak{g}/\mathcal{F}_i\mathfrak{g} \mid g_j \equiv g_k \pmod{\mathcal{F}_k\mathfrak{g}} \text{ for all } j > k\}.$$

Using the fact that $\mathcal{F}_k(\mathfrak{g})$ is an ideal of \mathfrak{g} , it is readily seen that $\widehat{\mathfrak{g}}$ is a Lie algebra, with Lie bracket defined componentwise. Furthermore, $\widehat{\mathfrak{g}}$ has a natural inverse limit filtration, $\widehat{\mathcal{F}}$, given by

$$(3) \quad \widehat{\mathcal{F}}_k\widehat{\mathfrak{g}} := \widehat{\mathcal{F}}_k\mathfrak{g} = \varprojlim_{i \geq k} \mathcal{F}_k\mathfrak{g}/\mathcal{F}_i\mathfrak{g} = \{(g_1, g_2, \dots) \in \widehat{\mathfrak{g}} \mid g_i = 0 \text{ for all } i < k\}.$$

Note that $\widehat{\mathcal{F}_k \mathfrak{g}} = \overline{\mathcal{F}_k \mathfrak{g}}$, and so each term of the filtration $\widehat{\mathcal{F}}$ is a closed Lie ideal of $\widehat{\mathfrak{g}}$. Furthermore, the Lie algebra $\widehat{\mathfrak{g}}$, endowed with this filtration, is both complete and separated.

Let $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ be the canonical map to the completion. Then ι is a morphism of Lie algebras, preserving the respective filtrations. Clearly, $\ker(\iota) = \bigcap_{k \geq 1} \mathcal{F}_k \mathfrak{g}$. Hence, ι is injective if and only if \mathfrak{g} is separated. Furthermore, ι is surjective if and only if \mathfrak{g} is complete.

2.4. Filtered Lie algebras. A *filtered Lie algebra* (over the field \mathbb{k}) is a Lie algebra \mathfrak{g} endowed with a \mathbb{k} -vector filtration $\{\mathcal{F}_k \mathfrak{g}\}_{k \geq 1}$ satisfying the ‘multiplicativity’ condition

$$(4) \quad [\mathcal{F}_r \mathfrak{g}, \mathcal{F}_s \mathfrak{g}] \subseteq \mathcal{F}_{r+s} \mathfrak{g}$$

for all $r, s \geq 1$. Obviously, this condition implies that each subspace $\mathcal{F}_k \mathfrak{g}$ is a Lie ideal, and so, in particular, \mathcal{F} is a Lie algebra filtration. Let

$$(5) \quad \text{gr}^{\mathcal{F}}(\mathfrak{g}) := \bigoplus_{k \geq 1} \mathcal{F}_k \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g}.$$

be the associated graded vector space to the filtration \mathcal{F} on \mathfrak{g} . Condition (4) implies that the Lie bracket map on \mathfrak{g} descends to a map $[\cdot, \cdot]: \text{gr}^{\mathcal{F}}(\mathfrak{g}) \times \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{F}}(\mathfrak{g})$, which makes $\text{gr}^{\mathcal{F}}(\mathfrak{g})$ into a graded Lie algebra, with graded pieces given by decomposition (5).

A morphism of filtered Lie algebras is a linear map $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ preserving Lie brackets and the given filtrations, \mathcal{F} and \mathcal{G} . Such a map induces morphisms between nilpotent quotients, $\phi_k: \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$, and a morphism of associated graded Lie algebras, $\text{gr}(\phi): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{G}}(\mathfrak{h})$.

If \mathfrak{g} is a filtered Lie algebra with a multiplicative filtration \mathcal{F} , then its completion, $\widehat{\mathfrak{g}}$, is again a filtered Lie algebra with the completed multiplicative filtration $\widehat{\mathcal{F}}$. By construction, the canonical map to the completion, $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$, is a morphism of filtered Lie algebras. It is readily seen that the induced morphism, $\text{gr}(\iota): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\widehat{\mathcal{F}}}(\widehat{\mathfrak{g}})$, is an isomorphism. Moreover, if \mathfrak{g} is both complete and separated, then the map $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ itself is an isomorphism of filtered Lie algebras.

Lemma 2.1. *Let $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ be a morphism of complete, separated, filtered Lie algebras, and suppose $\text{gr}(\phi): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{G}}(\mathfrak{h})$ is an isomorphism. Then ϕ is also an isomorphism.*

Proof. By assumption, the homomorphisms $\text{gr}_k(\phi): \mathcal{F}_k \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathcal{G}_k \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$ are isomorphisms, for all $k \geq 1$. An easy induction on k shows that all maps $\phi_k: \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$ are isomorphisms. Therefore, the map $\hat{\phi}: \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{h}}$ is an isomorphism. On the other hand, both \mathfrak{g} and \mathfrak{h} are complete and separated, and so $\mathfrak{g} = \widehat{\mathfrak{g}}$ and $\mathfrak{h} = \widehat{\mathfrak{h}}$. Hence $\phi = \hat{\phi}$, and we are done. \square

2.5. The degree completion. Any Lie algebra \mathfrak{g} comes equipped with a lower central series (LCS) filtration, $\{\Gamma_k(\mathfrak{g})\}_{k \geq 1}$, defined by $\Gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma_k(\mathfrak{g}) = [\Gamma_{k-1}(\mathfrak{g}), \mathfrak{g}]$ for $k \geq 2$. Clearly, this is a multiplicative filtration. Any other such filtration $\{\mathcal{F}_k(\mathfrak{g})\}_{k \geq 1}$ on \mathfrak{g} is coarser than this filtration; that is, $\Gamma_k \mathfrak{g} \subseteq \mathcal{F}_k \mathfrak{g}$, for all $k \geq 1$. Any Lie algebra morphism $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ preserves LCS filtrations. Furthermore, the quotient Lie algebras $\mathfrak{g} / \Gamma_k \mathfrak{g}$ are nilpotent. For simplicity, we shall write $\text{gr}(\mathfrak{g}) := \text{gr}^{\Gamma}(\mathfrak{g})$ for the associated graded Lie algebra and $\widehat{\mathfrak{g}}$ for the completion of \mathfrak{g} with respect to the LCS filtration Γ . Furthermore, we shall take $\widehat{\Gamma}_k = \overline{\Gamma}_k$ as the canonical filtration on $\widehat{\mathfrak{g}}$.

Every graded Lie algebra, $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$, has a canonical decreasing filtration induced by the grading, $\mathcal{F}_k \mathfrak{g} = \bigoplus_{i \geq k} \mathfrak{g}_i$. Moreover, if \mathfrak{g} is generated in degree 1, then this filtration coincides with the LCS filtration $\Gamma_k(\mathfrak{g})$. In particular, the associated graded Lie algebra with respect to \mathcal{F} coincides with \mathfrak{g} . In this case, the completion of \mathfrak{g} with respect to the lower central series (or, degree) filtration is called the *degree completion* of \mathfrak{g} , and is simply denoted by $\widehat{\mathfrak{g}}$. It is readily seen that $\widehat{\mathfrak{g}} \cong \prod_{i \geq 1} \mathfrak{g}_i$.

Therefore, the morphism $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ is injective, and induces an isomorphism $\mathfrak{g} \cong \text{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{g}})$. Moreover, if \mathfrak{h} is a graded Lie subalgebra of \mathfrak{g} , then $\widehat{\mathfrak{h}} = \overline{\mathfrak{h}}$ and

$$(6) \quad \text{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{h}}) = \mathfrak{h}.$$

Lemma 2.2. *If \mathfrak{Q} is a free Lie algebra generated in degree 1, and \mathfrak{r} is a homogeneous ideal, then the projection $\pi: \mathfrak{Q} \rightarrow \mathfrak{Q}/\mathfrak{r}$ induces an isomorphism $\widehat{\mathfrak{Q}}/\overline{\mathfrak{r}} \xrightarrow{\cong} \widehat{\mathfrak{Q}/\mathfrak{r}}$.*

Proof. Without loss of generality, we may assume that $\mathfrak{r} \subset [\mathfrak{Q}, \mathfrak{Q}]$. The projection $\pi: \mathfrak{Q} \rightarrow \mathfrak{Q}/\mathfrak{r}$ extends to an epimorphism between the degree completions, $\widehat{\pi}: \widehat{\mathfrak{Q}} \rightarrow \widehat{\mathfrak{Q}/\mathfrak{r}}$. This morphism takes the ideal generated by \mathfrak{r} to 0; thus, by continuity, it induces an epimorphism of complete, filtered Lie algebras, $\widehat{\mathfrak{Q}}/\overline{\mathfrak{r}} \twoheadrightarrow \widehat{\mathfrak{Q}/\mathfrak{r}}$. Taking associated graded, we obtain an epimorphism $\text{gr}(\widehat{\pi}): \text{gr}(\widehat{\mathfrak{Q}}/\overline{\mathfrak{r}}) \twoheadrightarrow \text{gr}(\widehat{\mathfrak{Q}/\mathfrak{r}}) = \mathfrak{Q}/\mathfrak{r}$. This epimorphism admits a splitting, induced by the maps $\Gamma_n \mathfrak{Q} + \mathfrak{r} \rightarrow \widehat{\Gamma}_n \widehat{\mathfrak{Q}} + \overline{\mathfrak{r}}$; thus, $\text{gr}(\widehat{\pi})$ is an isomorphism. The claim now follows from Lemma 2.1. \square

2.6. Filtered-formality. We now consider in more detail the relationship between a filtered Lie algebra \mathfrak{g} and the completion of its associated graded Lie algebra, $\widehat{\text{gr}}(\mathfrak{g})$, endowed with the inverse limit filtration. Note that both Lie algebras share the same associated graded Lie algebra, namely, $\text{gr}(\mathfrak{g})$. In general, though, \mathfrak{g} may not be isomorphic to $\widehat{\text{gr}}(\mathfrak{g})$. Of course, this happens if \mathfrak{g} is not complete or separated, but it may happen even in the case when \mathfrak{g} is a (finite-dimensional) nilpotent Lie algebra. We shall illustrate this point in Examples 9.5 and 9.6 below.

The following definition will play a key role in the sequel.

Definition 2.3. A complete, separated, filtered Lie algebra \mathfrak{g} is *filtered-formal* if there is a filtered Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\text{gr}}(\mathfrak{g})$ which induces the identity on associated graded Lie algebras.

This notion appears in the work of Bezrukavnikov [4] and Hain [31], as well as in the work of Calaque et. al [7] under the name of ‘formality’, and in the work of Lee [45], under the name of ‘weak-formality’. The reasons for our choice of terminology will become more apparent in §6.

If \mathfrak{g} is a filtered-formal Lie algebra, there exists a graded Lie algebra \mathfrak{h} such that \mathfrak{g} is isomorphic to $\widehat{\mathfrak{h}} = \prod_{i \geq 1} \mathfrak{h}_i$. Conversely, if $\mathfrak{g} = \widehat{\mathfrak{h}}$ is the completion of a graded Lie algebra $\mathfrak{h} = \bigoplus_{i \geq 1} \mathfrak{h}_i$, then \mathfrak{g} is filtered-formal. Moreover, if \mathfrak{h} has homogeneous presentation $\mathfrak{h} = \text{lie}(V)/\mathfrak{r}$, with V finitely generated and concentrated in degree 1, then, by Lemma 2.2, the complete, filtered Lie algebra $\mathfrak{g} = \prod_{i \geq 1} \mathfrak{h}_i$ has presentation $\mathfrak{g} = \widehat{\text{lie}(V)/\mathfrak{r}}$.

Lemma 2.4. *Let \mathfrak{g} be a complete, separated, filtered Lie algebra. If there is a graded Lie algebra \mathfrak{h} and a Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\mathfrak{h}}$ preserving filtrations, then \mathfrak{g} is filtered-formal.*

Proof. By assumption, there exists a filtered Lie algebra isomorphism $\phi: \mathfrak{g} \rightarrow \widehat{\mathfrak{h}}$. The map ϕ induces an isomorphism of graded Lie algebras, $\text{gr}(\phi): \text{gr}(\mathfrak{g}) \rightarrow \mathfrak{h}$. In turn, the map $\psi := (\text{gr}(\phi))^{-1}$ induces an isomorphism $\widehat{\psi}: \widehat{\mathfrak{h}} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$ of completed Lie algebras. Hence, the composite $\tilde{\phi} := \widehat{\psi} \circ \phi: \mathfrak{g} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$ is an isomorphism of filtered Lie algebras inducing the identity on $\text{gr}(\mathfrak{g})$. The conclusion follows from Lemma 2.1. \square

Corollary 2.5. *Let \mathfrak{g} be a complete, separated, filtered Lie algebra, and suppose the associated graded Lie algebra $\text{gr}(\mathfrak{g})$ is generated in degree 1. Furthermore, suppose there is a morphism of filtered Lie algebras, $\phi: \mathfrak{g} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$, such that $\text{gr}_1(\phi)$ is an isomorphism. Then \mathfrak{g} is filtered-formal.*

Proof. Consider the morphism $\text{gr}(\phi): \text{gr}(\mathfrak{g}) \rightarrow \text{gr}(\mathfrak{g})$. Since $\text{gr}(\mathfrak{g})$ is generated in degree 1, and since $\text{gr}_1(\phi)$ is an isomorphism, the map $\text{gr}(\phi)$ is an isomorphism. By Lemma 2.1, the map ϕ itself is an isomorphism. The conclusion follows from Lemma 2.4. \square

2.7. Descent of filtered-formality. We now show that filtered-formality is compatible with extension of scalars, and, more importantly, that filtered-formality enjoys a descent property, under some mild finiteness assumptions. As usual, all the ground fields will be of characteristic 0. We start with an easy lemma, which follows from the fact that completion commutes with tensor products.

Lemma 2.6. *Let \mathfrak{g} be a filtered-formal \mathbb{k} -Lie algebra, and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Then the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is also filtered-formal.*

The proof of the next result is based on recent work of Enriquez [20] and Maassarani [49]. The key tool is Proposition 7.6 from [20], which in turn was inspired by work of Drinfeld [17]. The structure of the proof follows to a large extent the approach from [49], where a particular example (the Malcev Lie algebra of the fundamental group of the orbit configuration space of a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ acting on \mathbb{CP}^1) is treated.

Theorem 2.7. *Let \mathfrak{g} be a complete, separated, filtered \mathbb{k} -Lie algebra such that $\mathrm{gr}(\mathfrak{g})$ is finitely generated in degree 1. If $\mathbb{k} \subset \mathbb{K}$ is a field extension, then \mathfrak{g} is filtered-formal if and only if the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal.*

Proof. The forward implication follows at once from Lemma 2.6. For the backward implication, suppose $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal over \mathbb{K} . Again in view of Lemma 2.6, we may assume without loss of generality that $\mathbb{k} = \mathbb{Q}$ and $\mathbb{K} = \overline{\mathbb{K}}$.

Set $\mathfrak{h} = \widehat{\mathrm{gr}}(\mathfrak{g})$, and let $\{\mathcal{G}_k\}_{k \geq 1}$ be the inverse limit filtration on \mathfrak{h} , coming from the degree filtration on $\mathrm{gr}(\mathfrak{g})$. For simplicity, let us write $\mathfrak{g}^i = \mathfrak{g}/\mathcal{F}_{i+1}\mathfrak{g}$ and $\mathfrak{h}^i = \mathfrak{h}/\mathcal{G}_{i+1}\mathfrak{h}$ for the respective quotient Lie algebras. As noted in [49, Lem. 6.1], the image of $\mathcal{F}_k(\mathfrak{g})$ in \mathfrak{g}^i is $\Gamma_k(\mathfrak{g}^i)$. In particular, \mathfrak{g}^1 is canonically isomorphic to the abelianizations of all \mathfrak{g}^i and of \mathfrak{g} . A similar statement holds for \mathfrak{h} .

For each $i \geq 1$, let $T_i = \mathrm{Iso}_1(\mathfrak{g}^i, \mathfrak{h}^i)$ be the affine \mathbb{Q} -scheme of filtration-preserving Lie algebra isomorphisms from \mathfrak{g}^i to \mathfrak{h}^i inducing the identity on abelianizations. As shown in [49, Prop. 6.2], these schemes form in natural way an inverse system; let $T = \varprojlim_i T_i$. Similarly, let $U_i = \mathrm{Aut}_1(\mathfrak{g}^i)$ be the unipotent \mathbb{Q} -group of automorphisms of \mathfrak{g}^i inducing the identity on abelianization, and let $U = \varprojlim_i U_i$ be the corresponding prounipotent \mathbb{Q} -group scheme. It is then readily seen that each U_i is a torsor under the natural left action of T_i , i.e., the action of $U_i(\mathbb{k})$ on $T_i(\mathbb{k})$ is free and transitive whenever $\mathbb{Q} \subset \mathbb{k}$ is a field extension such that $T_i(\mathbb{k})$ is non-empty. Furthermore, as noted in [49, Prop. 6.6], the U_i -actions on the torsors T_i are compatible with the canonical projections; thus, T is also a torsor under the action of U .

By assumption, $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{K}$ is filtered-formal. In view of Corollary 2.5, this condition is equivalent to the existence of a filtered Lie algebra isomorphism $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{K} \rightarrow \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{K}$ inducing the canonical identification $\mathfrak{g}^1 \otimes_{\mathbb{Q}} \mathbb{K} = \mathfrak{h}^1 \otimes_{\mathbb{Q}} \mathbb{K}$. That is, our assumption is equivalent to the fact that $T(\mathbb{K}) \neq \emptyset$. It remains to show that $T(\mathbb{Q}) \neq \emptyset$.

When $\mathbb{K} = \mathbb{C}$, this claim follows at once from Proposition 7.6 in [20]. The proof of that proposition involves two steps: first a descent from \mathbb{C} to $\overline{\mathbb{Q}}$, and then from $\overline{\mathbb{Q}}$ to \mathbb{Q} . To handle an arbitrary extension $\mathbb{Q} \subset \mathbb{K}$, we only need to modify the first step, and descend from $\mathbb{K} = \overline{\mathbb{K}}$ to $\overline{\mathbb{Q}}$. This is done by means of the same type of Hilbert's Nullstellensatz argument as the one sketched in [20]; we refer to the proof of [58, Cor. 5.8] for more details on how such an argument works. \square

As we shall see in Proposition 9.3, the above theorem generalizes a recent result of Cornulier (Theorem 3.14 from [13]).

2.8. Products and coproducts. The category of Lie algebras admits both products and coproducts. We end this section by showing that filtered-formality behaves well with respect to these operations.

Lemma 2.8. *Let \mathfrak{m} and \mathfrak{n} be two filtered-formal Lie algebras. Then $\mathfrak{m} \times \mathfrak{n}$ is also filtered-formal.*

Proof. By assumption, there exist graded Lie algebras \mathfrak{g} and \mathfrak{h} such that $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{i \geq 1} \mathfrak{g}_i$ and $\mathfrak{n} \cong \widehat{\mathfrak{h}} = \prod_{i \geq 1} \mathfrak{h}_i$. Then $\mathfrak{m} \times \mathfrak{n}$ is isomorphic to $(\prod_{i \geq 1} \mathfrak{g}_i) \times (\prod_{i \geq 1} \mathfrak{h}_i) = \prod_{i \geq 1} (\mathfrak{g}_i \times \mathfrak{h}_i) = \widehat{\mathfrak{g} \times \mathfrak{h}}$. Hence, $\mathfrak{m} \times \mathfrak{n}$ is filtered-formal. \square

Now let $*$ denote the usual coproduct (or, free product) of Lie algebras, and let $\widehat{*}$ be the coproduct in the category of complete, filtered Lie algebras. By definition,

$$(7) \quad \mathfrak{m} \widehat{*} \mathfrak{n} = \widehat{\mathfrak{m} * \mathfrak{n}} = \varprojlim_k (\mathfrak{m} * \mathfrak{n}) / \Gamma_k(\mathfrak{m} * \mathfrak{n}).$$

We refer to Lazarev and Markl [44] for a detailed study of this notion.

Lemma 2.9. *Let \mathfrak{m} and \mathfrak{n} be two filtered-formal Lie algebras. Then $\mathfrak{m} \widehat{*} \mathfrak{n}$ is also filtered-formal.*

Proof. As before, write $\mathfrak{m} = \widehat{\mathfrak{g}}$ and $\mathfrak{n} = \widehat{\mathfrak{h}}$, for some graded Lie algebras \mathfrak{g} and \mathfrak{h} . The canonical inclusions $\alpha: \mathfrak{g} \hookrightarrow \mathfrak{m}$ and $\beta: \mathfrak{h} \hookrightarrow \mathfrak{n}$ induce a monomorphism of filtered Lie algebras, $\widehat{\alpha * \beta}: \widehat{\mathfrak{g} * \mathfrak{h}} \rightarrow \widehat{\mathfrak{m} * \mathfrak{n}}$. Using [44, (9.3)], we infer that the induced morphism between associated graded Lie algebras, $\text{gr}(\widehat{\alpha * \beta}): \text{gr}(\widehat{\mathfrak{g} * \mathfrak{h}}) \rightarrow \text{gr}(\widehat{\mathfrak{m} * \mathfrak{n}})$, is an isomorphism. Lemma 2.1 now implies that $\widehat{\alpha * \beta}$ is an isomorphism of filtered Lie algebras, thereby verifying the filtered-formality of $\mathfrak{m} \widehat{*} \mathfrak{n}$. \square

3. GRADED ALGEBRAS AND KOSZUL DUALITY

The notions of graded and filtered algebras are defined completely analogously for an (associative) algebra A : the multiplication map is required to preserve the grading, respectively the filtration on A . In this section we discuss several relationships between Lie algebras and associative algebras, focussing on the notion of quadratic and Koszul algebras.

3.1. Universal enveloping algebras. Given a Lie algebra \mathfrak{g} over a field \mathbb{k} of characteristic 0, let $U(\mathfrak{g})$ be its universal enveloping algebra. This is the filtered algebra obtained as the quotient of the tensor algebra $T(\mathfrak{g})$ by the (two-sided) ideal I generated by all elements of the form $a \otimes b - b \otimes a - [a, b]$ with $a, b \in \mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, the canonical map $\iota: \mathfrak{g} \rightarrow U(\mathfrak{g})$ is an injection, and the induced map, $\text{Sym}(\mathfrak{g}) \rightarrow \text{gr}(U(\mathfrak{g}))$, is an isomorphism of graded (commutative) algebras. In this section, all tensor products are over \mathbb{k} .

Now suppose \mathfrak{g} is a finitely generated, graded Lie algebra. Then $U(\mathfrak{g})$ is isomorphic (as a graded vector space) to a polynomial algebra in variables indexed by bases for the graded pieces of \mathfrak{g} , with degrees set accordingly. Hence, its Hilbert series is given by

$$(8) \quad \text{Hilb}(U(\mathfrak{g}), t) = \prod_{i \geq 1} (1 - t^i)^{-\dim(\mathfrak{g}_i)}.$$

For instance, if $\mathfrak{g} = \text{lie}(V)$ is the free Lie algebra on a finite-dimensional vector space V with all generators in degree 1, then $\dim(\mathfrak{g}_i) = \frac{1}{i} \sum_{d|i} \mu(d) \cdot n^{i/d}$, where $n = \dim V$ and $\mu: \mathbb{N} \rightarrow \{-1, 0, 1\}$ is the Möbius function.

Finally, suppose $\mathfrak{g} = \text{lie}(V)/\mathfrak{r}$ is a finitely presented, graded Lie algebra, with generators in degree 1 and relation ideal \mathfrak{r} generated by homogeneous elements g_1, \dots, g_m . Then $U(\mathfrak{g})$ is the quotient of $T(V)$ by the two-sided ideal generated by $\iota(g_1), \dots, \iota(g_m)$, where $\iota: \text{lie}(V) \hookrightarrow T(V)$ is the canonical inclusion. In particular, if \mathfrak{g} is a quadratic Lie algebra, then $U(\mathfrak{g})$ is a quadratic algebra.

3.2. Quadratic algebras. Now let A be a graded \mathbb{k} -algebra. We will assume throughout that A is non-negatively graded, i.e., $A = \bigoplus_{i \geq 0} A_i$, and connected, i.e., $A_0 = \mathbb{k}$. Every such algebra may be realized as the quotient of a tensor algebra $T(V)$ by a homogeneous, two-sided ideal I . We will further assume that $\dim V < \infty$.

An algebra A as above is said to be *quadratic* if $A_1 = V$ and the ideal I is generated in degree 2, i.e., $I = \langle I_2 \rangle$, where $I_2 = I \cap (V \otimes V)$. Given a quadratic algebra $A = T(V)/I$, identify $V^* \otimes V^* \cong (V \otimes V)^*$, and define the *quadratic dual* of A to be the algebra $A^1 = T(V^*)/I^\perp$, where $I^\perp \subset T(V^*)$ is the ideal generated by the vector subspace $I_2^\perp := \{\alpha \in V^* \otimes V^* \mid \alpha(I_2) = 0\}$. Clearly, A^1 is also a quadratic algebra, and $(A^1)^1 = A$. For any graded algebra $A = T(V)/I$, we can define its quadrature closure as $\text{q}A = T(V)/\langle I_2 \rangle$. For more details on all this, we refer to [66].

Proposition 3.1. *Let \mathfrak{g} be a finitely generated graded Lie algebra generated in degree 1. There is then a unique, functorially defined quadratic Lie algebra, $\text{q}\mathfrak{g}$, such that $U(\text{q}\mathfrak{g}) = \text{q}U(\mathfrak{g})$.*

Proof. Suppose \mathfrak{g} has presentation $\text{lie}(V)/\mathfrak{r}$. Then $U(\mathfrak{g})$ has a presentation $T(V)/\langle \iota(\mathfrak{r}) \rangle$. Set $\text{q}\mathfrak{g} = \text{lie}(V)/\langle \mathfrak{r}_2 \rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \text{lie}_2(V)$; then $U(\text{q}\mathfrak{g})$ has presentation $T(V)/\langle \iota(\mathfrak{r}_2) \rangle$. One can see that $\iota(\mathfrak{r}_2) = \iota(\mathfrak{r}) \cap V \otimes V$. \square

A *commutative graded algebra* (for short, a *cGA*) is a graded \mathbb{k} -algebra as above, which in addition is graded-commutative, i.e., if $a \in A_i$ and $b \in A_j$, then $ab = (-1)^{ij}ba$. If all generators of A are in degree 1, then A can be written as $A = \wedge(V)/J$, where $\wedge(V)$ is the exterior algebra on the \mathbb{k} -vector space $V = A_1$, and J is a homogeneous ideal in $\wedge(V)$ with $J_1 = 0$. If, furthermore, J is generated in degree 2, then A is a quadratic *cGA*. The next lemma follows straight from the definitions.

Lemma 3.2. *Let $W \subset V \wedge V$ be a linear subspace, and let $A = \wedge(V)/\langle W \rangle$ be the corresponding quadratic *cGA*. Then $A^1 = T(V^*)/\langle \iota(W^\vee) \rangle$, where*

$$(9) \quad W^\vee := \{\alpha \in V^* \wedge V^* \mid \alpha(W) = 0\} = W^\perp \cap (V^* \wedge V^*),$$

and $\iota: V^* \wedge V^* \hookrightarrow V^* \otimes V^*$ is the inclusion map, given by $x \wedge y \mapsto x \otimes y - y \otimes x$.

For instance, if $A = \wedge(V)$, then $A^1 = \text{Sym}(V^*)$. Likewise, if $A = \wedge(V)/\langle V \wedge V \rangle = \mathbb{k} \oplus V$, then $A^1 = T(V^*)$.

3.3. Holonomy Lie algebras. Let A be a graded, graded-commutative algebra. Recall we are assuming that $A_0 = \mathbb{k}$ and $\dim A_1 < \infty$. Because of graded-commutativity, the multiplication map $A_1 \otimes A_1 \rightarrow A_2$ factors through a linear map $\mu_A: A_1 \wedge A_1 \rightarrow A_2$. Dualizing this map, and identifying $(A_1 \wedge A_1)^* \cong A_1^* \wedge A_1^*$, we obtain a linear map, $\partial_A = (\mu_A)^*: A_2^* \rightarrow A_1^* \wedge A_1^*$. Finally, identify $A_1^* \wedge A_1^*$ with $\text{lie}_2(A_1^*)$ via the map $x \wedge y \mapsto [x, y]$.

Definition 3.3. The *holonomy Lie algebra* of A is the quotient

$$(10) \quad \mathfrak{h}(A) = \text{lie}(A_1^*)/\langle \text{im}(\partial_A) \rangle$$

of the free Lie algebra on A_1^* by the ideal generated by the image of ∂_A under the above identification. Alternatively, using the notation from (9), we have that

$$(11) \quad \mathfrak{h}(A) = \text{lie}(A_1^*)/\langle \ker(\mu_A)^\vee \rangle.$$

Plainly, $\mathfrak{h}(A)$ is a quadratic Lie algebra. This construction is functorial: if $\varphi: A \rightarrow B$ is a morphism of *cGAs* as above, the induced map, $\text{lie}(\varphi_1^*): \text{lie}(B_1^*) \rightarrow \text{lie}(A_1^*)$, factors through a morphism of graded Lie algebras, $\mathfrak{h}(\varphi): \mathfrak{h}(B) \rightarrow \mathfrak{h}(A)$; moreover, if φ is injective, then $\mathfrak{h}(\varphi)$ is surjective. The Lie algebra

$\mathfrak{h}(A)$ depends only on information encoded in the map $\mu_A: A_1 \wedge A_1 \rightarrow A_2$. More precisely, let qA be the quadratic closure of A ; then

$$(12) \quad qA = \wedge(A_1)/\langle K \rangle,$$

where $K = \ker(\mu_A) \subset A_1 \wedge A_1$. Then qA is a commutative, quadratic algebra, which comes equipped with a canonical morphism $qA \rightarrow A$, which is an isomorphism in degree 1 and is injective in degree 2. It is readily verified that the induced morphism between holonomy Lie algebras, $\mathfrak{h}(A) \rightarrow \mathfrak{h}(qA)$, is an isomorphism.

The following proposition is a slight generalization of a result of Papadima–Yuzvinsky ([64, Lem. 4.1]).

Proposition 3.4. *Let A be a commutative graded algebra. Then $U(\mathfrak{h}(A))$ is a quadratic algebra, and $U(\mathfrak{h}(A)) = (qA)^\dagger$.*

Proof. By the above, $qA = \wedge(A_1)/\langle K \rangle$, where $K = \ker(\mu_A)$. On the other hand, by (11) we have that $\mathfrak{h}(A) = \text{lie}(A_1^*)/\langle K^\vee \rangle$. Hence, by Lemma 3.2, $U(\mathfrak{h}(A)) = T(V^*)/\langle \iota(K^\vee) \rangle = (qA)^\dagger$. \square

Combining Propositions 3.1 and 3.4, we obtain the following corollary, which expresses the quadratic closure of a Lie algebra as the holonomy Lie algebra of a certain quadratic algebra.

Corollary 3.5. *Let \mathfrak{g} be a finitely generated graded Lie algebra generated in degree 1. Then $\mathfrak{h}((qU(\mathfrak{g}))^\dagger) = \mathfrak{qg}$.*

Work of Löfwall [48, Thm. 1.1] yields another interpretation of the universal enveloping algebra of the holonomy Lie algebra.

Proposition 3.6 ([48]). *Let $[\text{Ext}_A^1(\mathbb{k}, \mathbb{k})] := \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$ be the linear strand in the Yoneda algebra of A . Then $U(\mathfrak{h}(A)) \cong [\text{Ext}_A^1(\mathbb{k}, \mathbb{k})]$.*

In particular, the graded ranks of $\mathfrak{h} = \mathfrak{h}(A)$ are given by $\prod_{n \geq 1} (1 - t^n)^{\dim \mathfrak{h}_n} = \sum_{i \geq 0} b_{ii} t^i$, where $b_{ii} = \dim_{\mathbb{k}} \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$. The next proposition shows that every quadratic Lie algebra can be realized as the holonomy Lie algebra of a (quadratic) algebra.

Proposition 3.7. *Let \mathfrak{g} be a quadratic Lie algebra. There is then a commutative quadratic algebra A such that $\mathfrak{g} = \mathfrak{h}(A)$.*

Proof. By assumption, $\mathfrak{g} = \text{lie}(V)/\langle W \rangle$, where W is a linear subspace of $V \wedge V$. Define $A = \wedge(V^*)/\langle W^\vee \rangle$. Then, by (11), $\mathfrak{h}(A) = \text{lie}((V^*)^*)/\langle (W^\vee)^\vee \rangle = \text{lie}(V)/\langle W \rangle$. \square

3.4. Koszul algebras. Given any connected, locally finite, graded algebra A , the trivial A -module \mathbb{k} has a free, graded A -resolution of the form $\cdots \xrightarrow{\varphi_3} A^{n_2} \xrightarrow{\varphi_2} A^{n_1} \xrightarrow{\varphi_1} A \rightarrow \mathbb{k}$; such a resolution is *minimal* if all the nonzero entries of the matrices φ_i have positive degrees. The algebra A is *Koszul* if the minimal A -resolution of \mathbb{k} is linear, or, equivalently, $\text{Ext}_A(\mathbb{k}, \mathbb{k}) = [\text{Ext}_A^1(\mathbb{k}, \mathbb{k})]$. Such an algebra is always quadratic, but the converse is far from true. If A is a Koszul algebra, then the quadratic dual A^\dagger is also a Koszul algebra, and the following ‘Koszul duality’ formula holds:

$$(13) \quad \text{Hilb}(A, t) \cdot \text{Hilb}(A^\dagger, -t) = 1.$$

Furthermore, if A is a graded algebra of the form $A = T(V)/I$, where I is an ideal admitting a (noncommutative) quadratic Gröbner basis, then A is a Koszul algebra (see [28]).

Corollary 3.8. *Let A be a connected, commutative graded algebra. If qA is a Koszul algebra, then $\text{Hilb}(qA, -t) \cdot \text{Hilb}(U(\mathfrak{b}(A)), t) = 1$.*

Example 3.9. Consider the quadratic algebra $A = \wedge(u_1, u_2, u_3, u_4)/(u_1u_2 - u_3u_4)$. Clearly, we have $\text{Hilb}(A, t) = 1 + 4t + 5t^2$. If A were Koszul, then formula (13) would give $\text{Hilb}(A^!, t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 + \dots$, which is impossible.

Example 3.10. The quasitriangular Lie algebra qtr_n defined in [1] is generated by x_{ij} , $1 \leq i \neq j \leq n$ with relations $[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0$ for distinct i, j, k and $[x_{ij}, x_{kl}] = 0$ for distinct i, j, k, l . The Lie algebra tr_n is the quotient Lie algebra of qtr_n by the ideal generated by $x_{ij} + x_{ji}$ for distinct $i \neq j$. In [1], Bartholdi et al. show that the quadratic dual algebras $U(\text{qtr}_n)^!$ and $U(\text{tr}_n)^!$ are Koszul, and compute their Hilbert series. They also state that neither qtr_n nor tr_n is filtered-formal for $n \geq 4$, and sketch a proof of this assertion; we give a detailed proof of this fact in [75].

4. MINIMAL MODELS AND (PARTIAL) FORMALITY

In this section, we discuss two basic notions in non-simply-connected rational homotopy theory: the minimal model and the (partial) formality properties of a differential graded algebra.

4.1. Minimal models of DGAs. We follow the approach of Sullivan [81], Deligne et al. [15], and Morgan [57], as further developed in [22, 23, 29, 33, 39, 50]. We start with some basic algebraic notions.

Definition 4.1. A *differential graded algebra* (for short, a DGA) over a field \mathbb{k} of characteristic 0 is a graded \mathbb{k} -algebra $A^* = \bigoplus_{n \geq 0} A^n$ equipped with a differential $d: A \rightarrow A$ of degree 1 satisfying $ab = (-1)^{mn}ba$ and $d(ab) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b)$ for any $a \in A^m$ and $b \in A^n$. We denote the DGA by (A^*, d) or simply by A^* if there is no confusion.

A morphism $f: A^* \rightarrow B^*$ between two DGAs is a degree zero algebra map which commutes with the differentials. A *Hirsch extension* (of degree i) is a DGA inclusion $\alpha: (A^*, d_A) \hookrightarrow (A^* \otimes \wedge(V), d)$, where V is a \mathbb{k} -vector space concentrated in degree i , while $\wedge(V)$ is the free graded-commutative algebra generated by V , and d sends V into A^{i+1} . We say this is a *finite Hirsch extension* if $\dim V < \infty$. Note that all tensor product in this section is over \mathbb{k} , hence we will denote \otimes for short.

Definition 4.2 ([81]). A DGA (A^*, d) is called *minimal* if $A^0 = \mathbb{k}$, and the following conditions are satisfied:

- (1) $A^* = \bigcup_{j \geq 0} A_j^*$, where $A_0 = \mathbb{k}$, and A_j is a Hirsch extension of A_{j-1} , for all $j \geq 0$.
- (2) The differential is *decomposable*, i.e., $dA^* \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{i \geq 1} A^i$.

The first condition implies that A^* has an increasing, exhausting filtration by the sub-DGAs A_j^* ; equivalently, A^* is free as a graded-commutative algebra on generators of degree ≥ 1 . (Note that we use the lower-index for the filtration, and the upper-index for the grading.) The second condition is automatically satisfied if A is generated in degree 1.

A DGA map $f: A \rightarrow B$ is said to be a *quasi-isomorphism* if all the induced maps in cohomology, $H^j(f): H^j(A) \rightarrow H^j(B)$, are isomorphisms. Two DGAs A and B are *weakly equivalent* (written $A \simeq B$) if there is a zig-zag of quasi-isomorphisms connecting them. Likewise, for an integer $i \geq 0$, a morphism $f: A \rightarrow B$ is an *i -quasi-isomorphism* if $H^j(f)$ is an isomorphism for $j \leq i$ and an injection for $j = i + 1$. Furthermore, A and B are *i -weakly equivalent* ($A \simeq_i B$) if there is a zig-zag of i -quasi-isomorphism connecting A to B . The next two lemmas follow straight from the definitions.

Lemma 4.3. *Any DGA morphism $\phi: (A, d_A) \rightarrow (B, d_B)$ extends to a DGA morphism of Hirsch extensions, $\bar{\phi}: (A, d_A) \otimes \wedge(x) \rightarrow (B, d_B) \otimes \wedge(y)$, provided that $d(y) = \phi(d(x))$. Moreover, if ϕ is a (quasi-) isomorphism, then so is $\bar{\phi}$.*

Lemma 4.4. *Let $\alpha: A \rightarrow B$ be the inclusion map of Hirsch extension of degree $i + 1$. Then α is an i -quasi-isomorphism.*

We say that a DGA B is a *minimal model* for a DGA A if B is a minimal DGA and there exists a quasi-isomorphism $f: B \rightarrow A$. Likewise, we say that a minimal DGA B is an *i -minimal model* for A if B is generated by elements of degree at most i , and there exists an i -quasi-isomorphism $f: B \rightarrow A$. A basic result in rational homotopy theory is the following existence and uniqueness theorem, first proved for (full) minimal models by Sullivan [81], and in full generality by Morgan in [57, Thm. 5.6].

Theorem 4.5 ([57, 81]). *Each connected DGA (A, d) has a minimal model $\mathcal{M}(A)$, unique up to isomorphism. Likewise, for each $i \geq 0$, there is an i -minimal model $\mathcal{M}(A, i)$, unique up to isomorphism.*

It follows from the proof of Theorem 4.5 that the minimal model $\mathcal{M}(A)$ is isomorphic to a minimal model built from the i -minimal model $\mathcal{M}(A, i)$ by means of Hirsch extensions in degrees $i + 1$ and higher. Thus, in view of Lemma 4.4, $\mathcal{M}(A, i) \simeq_i \mathcal{M}(A)$.

4.2. Minimal models and holonomy Lie algebras. Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal DGA over \mathbb{k} , generated in degree 1. Following [57, 39, 23], let us consider the filtration

$$(14) \quad \mathbb{k} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M} = \bigcup_i \mathcal{M}_i,$$

where \mathcal{M}_1 is the subalgebra of \mathcal{M} generated by $x \in \mathcal{M}^1$ such that $dx = 0$, and \mathcal{M}_i is the subalgebra of \mathcal{M} generated by $x \in \mathcal{M}^1$ such that $dx \in \mathcal{M}_{i-1}$ for $i > 1$. Each inclusion $\mathcal{M}_{i-1} \subset \mathcal{M}_i$ is a Hirsch extension of the form $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \wedge(V_i)$, where $V_i := \ker(H^2(\mathcal{M}_{i-1}) \rightarrow H^2(\mathcal{M}))$. Taking the degree 1 part of the filtration (14), we obtain the filtration $\mathbb{k} = \mathcal{M}_0^1 \subset \mathcal{M}_1^1 \subset \cdots \subset \mathcal{M}^1$.

Now assume each of the above Hirsch extensions is finite, i.e., $\dim(V_i) < \infty$ for all i . Using the fact that $d(V_i) \subset \mathcal{M}_{i-1}$, we see that each dual vector space $\mathfrak{L}_i = (\mathcal{M}_i^1)^*$ acquires the structure of a \mathbb{k} -Lie algebra by setting

$$(15) \quad \langle [u^*, v^*], w \rangle = \langle u^* \wedge v^*, dw \rangle$$

for $u, v, w \in \mathcal{M}_i^1$. Clearly, $d(V_1) = 0$, and thus $\mathfrak{L}_1 = (V_1)^*$ is an abelian Lie algebra. Using the vector space decompositions $\mathcal{M}_i^1 = \mathcal{M}_{i-1}^1 \oplus V_i$ and $\mathcal{M}_i^2 = \mathcal{M}_{i-1}^2 \oplus (\mathcal{M}_{i-1}^1 \otimes V_i) \oplus \wedge^2(V_i)$ we easily see that the canonical projection $\mathfrak{L}_i \rightarrow \mathfrak{L}_{i-1}$ (i.e., the dual of the inclusion map $\mathcal{M}_{i-1} \hookrightarrow \mathcal{M}_i$) has kernel V_i^* , and this kernel is central inside \mathfrak{L}_i . Therefore, we obtain a tower of finite-dimensional nilpotent \mathbb{k} -Lie algebras,

$$(16) \quad 0 \longleftarrow \mathfrak{L}_1 \longleftarrow \mathfrak{L}_2 \longleftarrow \cdots \longleftarrow \mathfrak{L}_i \longleftarrow \cdots .$$

The inverse limit of this tower, $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra such that $\mathfrak{L}/\widehat{\Gamma}_{i+1}\mathfrak{L} = \mathfrak{L}_i$, for each $i \geq 1$. Conversely, from a tower as in (16), we can construct a sequence of finite Hirsch extensions \mathcal{M}_i as in (14). Furthermore, the DGA \mathcal{M}_i , with differential defined by (15), coincides with the Chevalley–Eilenberg complex $(\wedge(\mathfrak{L}_i^*), d)$ associated to the finite-dimensional Lie algebra $\mathfrak{L}_i = \mathfrak{L}(\mathcal{M}_i)$, as in [23, §II] and [35, §VII]. In particular,

$$(17) \quad H^*(\mathcal{M}_i) \cong H^*(\mathfrak{L}_i; \mathbb{k}).$$

The direct limit of the above sequence of Hirsch extensions, $\mathcal{M} = \bigcup_i \mathcal{M}_i$, is a minimal \mathbb{k} -DGA generated in degree 1, which we denote by $\mathcal{M}(\mathfrak{L})$. We obtain in this fashion an adjoint correspondence

that sends \mathcal{M} to the pronilpotent Lie algebra $\mathfrak{L}(\mathcal{M})$ and conversely, sends a pronilpotent Lie algebra \mathfrak{L} to the minimal algebra $\mathcal{M}(\mathfrak{L})$. Under this correspondence, filtration-preserving DGA morphisms $\mathcal{M} \rightarrow \mathcal{N}$ get sent to filtration-preserving Lie morphisms $\mathfrak{L}(\mathcal{N}) \rightarrow \mathfrak{L}(\mathcal{M})$, and vice-versa.

4.3. Positive weights. Following [6, 57, 81], we say that a cGA A^* has *positive weights* if each graded piece has a vector space decomposition $A^i = \bigoplus_{\alpha \in \mathbb{Z}} A^{i,\alpha}$ with $A^{1,\alpha} = 0$ for $\alpha \leq 0$, such that $xy \in A^{i+j,\alpha+\beta}$ for $x \in A^{i,\alpha}$ and $y \in A^{j,\beta}$. Furthermore, we say that a DGA (A^*, d) has *positive weights* if the underlying cGA A^* has positive weights, and the differential is homogeneous with respect to those weights, i.e., $d(x) \in A^{i+1,\alpha}$ for $x \in A^{i,\alpha}$.

Now let (\mathcal{M}^*, d) be a minimal DGA generated in degree one, endowed with the canonical filtration $\{\mathcal{M}_i\}_{i \geq 0}$ constructed in (14), where each sub-DGA \mathcal{M}_i given by a Hirsch extension of the form $\mathcal{M}_{i-1} \otimes \wedge(V_i)$. The underlying cGA \mathcal{M}^* possesses a natural set of positive weights, which we will refer to as the *Hirsch weights*: simply declare V_i to have weight i , and extend those weights to \mathcal{M}^* multiplicatively. We say that the DGA (\mathcal{M}^*, d) has *positive Hirsch weights* if the differential d is homogeneous with respect to those weights. If this is the case, each sub-DGA \mathcal{M}_i also has positive Hirsch weights.

Lemma 4.6. *Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal DGA generated in degree one, with dual Lie algebra \mathfrak{L} . Then \mathcal{M} has positive Hirsch weights if and only if $\mathfrak{L} = \widehat{\text{gr}}(\mathfrak{L})$.*

Proof. As usual, write $\mathcal{M} = \bigcup \mathcal{M}_i$, with $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \wedge(V_i)$. Since \mathcal{M} is generated in degree 1, the differential is homogeneous with respect to the Hirsch weights if and only if $d(V_s) \subset \bigoplus_{i+j=s} V_i \wedge V_j$, for all $s \geq 1$. Passing now to the dual Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ and using (15), we see that this condition is equivalent to having $[V_i^*, V_j^*] \subset V_{i+j}^*$, for all $i, j \geq 1$. In turn, this is equivalent to saying that each \mathfrak{L}_i is a graded Lie algebra with $\text{gr}_k(\mathfrak{L}_i) = V_k^*$, for each $k \leq i$, which means that the filtered Lie algebra $\mathfrak{L} = \varprojlim_i \mathfrak{L}_i$ coincides with the completion of its associated graded Lie algebra, $\widehat{\text{gr}}(\mathfrak{L})$. \square

Remark 4.7. The property that the differential of \mathcal{M} be homogeneous with respect to the Hirsch weights is stronger than saying that the Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ is filtered-formal. The fact that this can happen is illustrated in Example 9.2.

Remark 4.8. If a minimal DGA is generated in degree 1 and has positive weights, but these weights do not coincide with the Hirsch weights, then the dual Lie algebra need not be filtered-formal. This phenomenon is illustrated in Example 9.5: there is a finitely generated nilpotent Lie algebra \mathfrak{m} for which the Chevalley–Eilenberg complex $\mathcal{M}(\mathfrak{m}) = \wedge(\mathfrak{m}^*)$ has positive weights, but those weights are not the Hirsch weights; moreover, \mathfrak{m} is not filtered-formal.

4.4. Dual Lie algebra and holonomy Lie algebra. Let (B^*, d) be a DGA, and let $A = H^*(B)$ be its cohomology algebra. Assume A is connected and $\dim A^1 < \infty$, and let $\mu: A^1 \wedge A^1 \rightarrow A^2$ be the multiplication map. By the discussion from §4.1, there is a 1-minimal model $\mathcal{M}(B, 1)$ for our DGA, unique up to isomorphism.

A concrete way to build such a model can be found in [15, 29, 57]. The first two steps of this construction are easy to describe. Set $V_1 = A^1$ and define $\mathcal{M}(B, 1)_1 = \wedge(V_1)$, with differential $d = 0$. Next, set $V_2 = \ker(\mu)$ and define $\mathcal{M}(B, 1)_2 = \wedge(V_1 \oplus V_2)$, with $d|_{V_2}$ equal to the inclusion map $V_2 \hookrightarrow A^1 \wedge A^1$. Let $\mathfrak{L}(B) = \mathfrak{L}(\mathcal{M}(B, 1))$ be the Lie algebra corresponding to the 1-minimal model of B . The next proposition, which generalizes a result of Kohno ([39, Lem. 4.9]), relates this Lie algebra to the holonomy Lie algebra $\mathfrak{h}(A)$ from Definition 3.3.

Proposition 4.9. *Let $\phi: \mathbf{L} \rightarrow \mathfrak{Q}(B)$ be the morphism defined by extending the identity map of V_1^* to the free Lie algebra $\mathbf{L} = \text{lie}(V_1^*)$, and let $J = \ker(\phi)$. There exists then an isomorphism of graded Lie algebras, $\mathfrak{h}(A) \cong \mathbf{L}/\langle J \cap \mathbf{L}_2 \rangle$, where $\mathfrak{h}(A)$ is the holonomy Lie algebra of $A = H^*(B)$.*

Proof. Let $\text{gr}(\phi): \mathbf{L} \rightarrow \text{gr}^{\widehat{\Gamma}}(\mathfrak{Q}(B))$ be the associated graded morphism of ϕ . Then the first graded piece, $\text{gr}_1(\phi): V_1^* \rightarrow V_1^*$, is the identity, while the second graded piece, $\text{gr}_2(\phi)$, can be identified with the Lie bracket map $V_1^* \wedge V_1^* \rightarrow V_2^*$, which is the dual of the differential $d: V_2 \rightarrow V_1 \wedge V_1$. From the construction of $\mathcal{M}(B, 1)_2$, there is an isomorphism $\ker(d^*) \cong \text{im}(\mu^*)$. Since $J \cap \mathbf{L}_2 = \ker(\text{gr}_2(\phi))$, we have that $\text{im}(\mu^*) = J \cap \mathbf{L}_2$, and the claim follows. \square

4.5. The completion of the holonomy Lie algebra. Let A^* be a commutative graded \mathbb{k} -algebra with $A^0 = \mathbb{k}$. Proceeding as above, by taking $B = A$ and $d = 0$ so that $H^*(B) = A$, we can construct a 1-minimal model $\mathcal{M} = \mathcal{M}(A, 1)$ for the algebra A in a ‘formal’ way, following the approach outlined by Carlson and Toledo in [8]. (A construction of the full, bigraded minimal model of a cGA can be found in [33, §3].)

As before, set $\mathcal{M}_1 = (\wedge(V_1), d = 0)$ where $V_1 = A^1$, and $\mathcal{M}_2 = (\wedge(V_1 \oplus V_2), d)$, where $V_2 = \ker(\mu: A^1 \wedge A^1 \rightarrow A^2)$ and $d: V_2 \hookrightarrow V_1 \wedge V_1$ is the inclusion map. After that, define inductively \mathcal{M}_i as $\mathcal{M}_{i-1} \otimes \wedge(V_i)$, where the vector space V_i fits into the short exact sequence

$$(18) \quad 0 \longrightarrow V_i \longrightarrow H^2(\mathcal{M}_{i-1}) \longrightarrow \text{im}(\mu) \longrightarrow 0,$$

while the differential d includes V_i into $V_1 \wedge V_{i-1} \subset \mathcal{M}_{i-1}$. In particular, the subalgebras \mathcal{M}_i constitute the canonical filtration (14) of \mathcal{M} , and the differential d preserves the Hirsch weights on \mathcal{M} . For these reasons, we call $\mathcal{M} = \mathcal{M}(A, 1)$ the *canonical* 1-minimal model of A .

The next theorem relates the Lie algebra dual to the canonical 1-minimal model of a cGA as above to its holonomy Lie algebra. A similar result was obtained by Markl and Papadima in [53]; see also Morgan [57, Thm. 9.4] and Remark 7.3.

Theorem 4.10. *Let A^* be a connected cGA with $\dim A^1 < \infty$. Let $\mathfrak{Q}(A) := \mathfrak{Q}(\mathcal{M}(A, 1))$ be the Lie algebra corresponding to the canonical 1-minimal model of A , and let $\mathfrak{h}(A)$ be the holonomy Lie algebra of A . There exists then an isomorphism of complete, filtered Lie algebras between $\mathfrak{Q}(A)$ and the degree completion $\widehat{\mathfrak{h}}(A)$.*

Proof. By Definition 3.3, the holonomy Lie algebra of A has presentation $\mathfrak{h}(A) = \mathbf{L}/\mathfrak{r}$, where $\mathbf{L} = \text{lie}(V_1^*)$ and \mathfrak{r} is the ideal generated by $\text{im}(\mu^*) \subset \mathbf{L}_2$. It follows that, for each $i \geq 1$, the nilpotent quotient $\mathfrak{h}_i(A) := \mathfrak{h}(A)/\Gamma_{i+1}\mathfrak{h}(A)$ has presentation $\mathbf{L}/(\mathfrak{r} + \Gamma_{i+1}\mathbf{L})$.

Consider now the dual Lie algebra $\mathfrak{Q}_i(A) = \mathfrak{Q}(\mathcal{M}_i)$. By construction, we have a vector space decomposition, $\mathfrak{Q}_i(A) = \bigoplus_{s \leq i} V_s^*$. The fact that $d(V_s) \subset V_1 \wedge V_{s-1}$ implies that the Lie bracket maps $V_1^* \wedge V_{s-1}^*$ onto V_s^* , for every $1 < s \leq i$. In turn, this implies that $\mathfrak{Q}_i(A)$ is an i -step nilpotent, graded Lie algebra generated in degree 1, with $\text{gr}_s(\mathfrak{Q}_i(A)) = V_s^*$ for $s \leq i$.

Let \mathfrak{r}_i be the kernel of the canonical projection $\pi_i: \mathbf{L} \rightarrow \mathfrak{Q}_i(A)$. By the Hopf formula, there is an isomorphism of graded vector spaces between $H_2(\mathfrak{Q}_i(A); \mathbb{k})$ and $\mathfrak{r}_i/[\mathbf{L}, \mathfrak{r}_i]$, the space of (minimal) generators for the homogeneous ideal \mathfrak{r}_i . On the other hand, $H^2(\mathcal{M}_i) \cong H^2(\mathfrak{Q}_i; \mathbb{k})$, by (17). Taking the dual of the exact sequence (18), we find that $H_2(\mathfrak{Q}_i(A); \mathbb{k}) \cong \text{im}(\mu^*) \oplus V_{i+1}^*$. We conclude that the ideal \mathfrak{r}_i is generated by $\text{im}(\mu^*)$ in degree 2 and a copy of V_{i+1}^* in degree $i + 1$.

Since $\text{gr}_2(\mathfrak{r}) = \text{im}(\mu^*)$, we infer that $\bigoplus_{s \leq i} \text{gr}_s(\mathfrak{r}_i) = \bigoplus_{s \leq i} \text{gr}_s(\mathfrak{r})$. Since $\mathfrak{Q}_i(A)$ is an i -step nilpotent Lie algebra, $\bigoplus_{s > i} \text{gr}_s(\mathfrak{r}_i) = \Gamma_{i+1}\mathbf{L}$. Therefore, $\Gamma_{i+1}\mathbf{L} + \mathfrak{r} = \mathfrak{r}_i$, and thus the identity map of \mathbf{L} induces an isomorphism $\mathfrak{Q}_i(A) \cong \mathfrak{h}_i(A)$, for each $i \geq 1$. Hence, $\mathfrak{Q}(A) \cong \widehat{\mathfrak{h}}(A)$, as filtered Lie algebras. \square

Corollary 4.11. *The graded ranks of the holonomy Lie algebra of a connected, graded algebra A are given by $\dim \mathfrak{h}_i(A) = \dim V_i$, where $\mathcal{M} = \wedge(\bigoplus_{i \geq 1} V_i)$ is the canonical 1-minimal model of $(A, d = 0)$.*

4.6. Partial formality and field extensions. The following notion, introduced by Sullivan in [81], and further developed in [15, 29, 50, 57], will play a central role in our study.

Definition 4.12. A DGA (A^*, d) over \mathbb{k} is said to be *formal* if there exists a quasi-isomorphism $\mathcal{M}(A) \rightarrow (H^*(A), d = 0)$. Likewise, (A^*, d) is said to be *i -formal* if there exists an i -quasi-isomorphism $\mathcal{M}(A, i) \rightarrow (H^*(A), d = 0)$.

In [50], Măcinic studies in detail these concepts. Evidently, if A is formal, then it is i -formal, for all $i \geq 0$, and, if A is i -formal, then it is j -formal for every $j \leq i$. Moreover, A is 0-formal if and only if $H^0(A) = \mathbb{k}$.

Lemma 4.13 ([50]). *A DGA (A^*, d) is i -formal if and only if (A^*, d) is i -weakly equivalent to $H^*(A)$ with zero differential.*

As a corollary, we deduce that i -formality is invariant under i -weakly equivalence.

Corollary 4.14. *Suppose $A \simeq_i B$. Then A is i -formal if and only if B is i -formal.*

Given a DGA (A, d) over a field \mathbb{k} of characteristic 0, and a field extension $\mathbb{k} \subset \mathbb{K}$, let $(A \otimes \mathbb{K}, d \otimes \text{id}_{\mathbb{K}})$ be the corresponding DGA over \mathbb{K} . (If the underlying field \mathbb{k} is understood, we will usually omit it from the tensor product $A \otimes_{\mathbb{k}} \mathbb{K}$.) The following result will be crucial for us in the sequel.

Theorem 4.15 (Thm. 6.8 in [33]). *Let (A^*, d_A) and (B^*, d_B) be two DGAs over \mathbb{k} whose cohomology algebras are connected and of finite type. Suppose there is an isomorphism of graded algebras, $f: H^*(A) \rightarrow H^*(B)$, and suppose $f \otimes \text{id}_{\mathbb{K}}: H^*(A) \otimes \mathbb{K} \rightarrow H^*(B) \otimes \mathbb{K}$ can be realized by a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ and $(B^* \otimes \mathbb{K}, d_B \otimes \text{id}_{\mathbb{K}})$. Then f can be realized by a weak equivalence between (A^*, d_A) and (B^*, d_B) .*

This theorem has an important corollary, based on the following lemma. For completeness, we provide proofs for these two results, which are stated without proof in [33].

Lemma 4.16 ([33]). *A DGA (A^*, d_A) with $H^*(A)$ of finite-type is formal if and only if the identity map of $H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$.*

Proof. The backwards implication is obvious. So assume (A^*, d_A) is formal, that is, there is a zig-zag of quasi-isomorphisms between (A^*, d_A) and $(H^*(A), d = 0)$. This yields an isomorphism in cohomology, $\phi: H^*(A) \rightarrow H^*(A)$. Composing the inverse of ϕ with the given zig-zag of quasi-isomorphisms defines a new weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$, which induces the identity map in cohomology. \square

Corollary 4.17 ([33]). *A \mathbb{k} -DGA (A^*, d_A) with $H^*(A)$ of finite-type is formal if and only if the \mathbb{K} -DGA $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ is formal.*

Proof. The forward implication is obvious. For the converse, suppose our \mathbb{K} -DGA is formal. By Lemma 4.16, there exists a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ and $(H^*(A) \otimes \mathbb{K}, d = 0)$ inducing the identity on $H^*(A) \otimes \mathbb{K}$. By Theorem 4.15, the map $\text{id}: H^*(A) \rightarrow H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$. That is, (A^*, d_A) is formal (over \mathbb{k}). \square

4.7. Field extensions and i -formality. We now use the aforementioned result of Halperin and Stasheff on full formality to establish an analogous result for partial formality. First we need an auxiliary construction, and a lemma.

Let $\mathcal{M}(A, i)$ be the i -minimal model of a DGA (A^*, d_A) . The degree $i + 1$ piece, $\mathcal{M}(A, i)^{i+1}$, is isomorphic to $(\ker(d^{i+1})) \oplus \mathcal{C}_{i+1}$, where $d^{i+1}: \mathcal{M}(A, i)^{i+1} \rightarrow \mathcal{M}(A, i)^{i+2}$ is the differential, and \mathcal{C}_{i+1} is a complement to its kernel. It is readily checked that the vector subspace $\mathcal{I}_i := \mathcal{C}_{i+1} \oplus \bigoplus_{s \geq i+2} \mathcal{M}(A, i)^s$ is an ideal of $\mathcal{M}(A, i)$, left invariant by the differential. Consider the quotient DGA, $\mathcal{M}[A, i] := \mathcal{M}(A, i)/\mathcal{I}_i$. Additively, we have that $\mathcal{M}[A, i] = \mathbb{k} \oplus \mathcal{M}(A, i)^1 \oplus \cdots \oplus \mathcal{M}(A, i)^i \oplus \ker(d^{i+1})$.

Lemma 4.18. *Suppose that $\dim H^{i+1}(\mathcal{M}(A, i)) < \infty$. The following statements are equivalent: (1) (A^*, d_A) is i -formal; (2) $\mathcal{M}(A, i)$ is i -formal; (3) $\mathcal{M}[A, i]$ is i -formal; and (4) $\mathcal{M}[A, i]$ is formal.*

Proof. Since $\mathcal{M}(A, i)$ is an i -minimal model for (A^*, d_A) , the two DGAs are i -quasi-isomorphic. The equivalence (1) \Leftrightarrow (2) follows from Corollary 4.14.

Now let $\psi: \mathcal{M}(A, i) \rightarrow \mathcal{M}[A, i]$ be the canonical projection. It is readily checked that the induced homomorphism, $H^j(\psi): H^j(\mathcal{M}(A, i)) \rightarrow H^j(\mathcal{M}[A, i])$, is an isomorphism in degrees up to and including $i + 1$. In particular, this shows that $\mathcal{M}(A, i)$ is an i -minimal model for $\mathcal{M}[A, i]$. The equivalence (2) \Leftrightarrow (3) again follows from Corollary 4.14.

Implication (4) \Rightarrow (3) is trivial, so it remains to establish (3) \Rightarrow (4). Assume the DGA $\mathcal{M}[A, i]$ is i -formal. Since $\mathcal{M}(A, i)$ is an i -minimal model for $\mathcal{M}[A, i]$, there is an i -quasi-isomorphism β as in diagram (19). In particular, the homomorphism, $H^{i+1}(\beta): H^{i+1}(\mathcal{M}(A, i)) \rightarrow H^{i+1}(\mathcal{M}[A, i])$, is injective. On the other hand, we know from the previous paragraph that $H^{i+1}(\mathcal{M}[A, i])$ and $H^{i+1}(\mathcal{M}(A, i))$ have the same dimension. Since by assumption $\dim H^{i+1}(\mathcal{M}(A, i)) < \infty$, we conclude that $H^{i+1}(\beta)$ is an isomorphism too.

$$(19) \quad \begin{array}{ccc} \mathcal{M}(A, i) & \xrightarrow{\beta} & (H^*(\mathcal{M}[A, i]), 0) \\ \psi \downarrow & \searrow \alpha & \uparrow \gamma \cong \\ \mathcal{M}[A, i] & \xleftarrow[\cong]{\phi} & \mathcal{M} \end{array}$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{M}[A, i])$ be the full minimal model of $\mathcal{M}[A, i]$. As mentioned right after Theorem 4.5, this model can be constructed by Hirsch extensions of degree $k \geq i + 1$, starting from the i -minimal model of $\mathcal{M}[A, i]$, which we can take to be $\mathcal{M}(A, i)$. Hence, the inclusion map, $\alpha: \mathcal{M}(A, i) \rightarrow \mathcal{M}$, induces isomorphisms in cohomology up to degree i , and a monomorphism in degree $i + 1$. Now, since $H^{i+1}(\mathcal{M})$ has the same dimension as $H^{i+1}(\mathcal{M}[A, i])$, and thus as $H^{i+1}(\mathcal{M}(A, i))$, the map $H(\alpha)$ is also an isomorphism in degree $i + 1$.

The DGA morphism β extends to a cga map $\gamma: \mathcal{M} \rightarrow H^*(\mathcal{M}[A, i])$ as in diagram (19), by sending the new generators to zero. Since the target of β vanishes in degrees $k \geq i + 2$ and has differential $d = 0$, the map γ is a DGA morphism. Furthermore, since $\gamma \circ \alpha = \beta$, we infer that γ induces isomorphisms in cohomology in degrees $k \leq i + 1$. Since $H^k(\mathcal{M}) = H^k(\mathcal{M}[A, i]) = 0$ for $k \geq i + 2$, we conclude that $H(\gamma)$ is an isomorphism in all degrees, i.e., γ is a quasi-isomorphism.

Finally, let $\phi: \mathcal{M} \rightarrow \mathcal{M}[A, i]$ be a quasi-isomorphism from the minimal model of $\mathcal{M}[A, i]$ to this DGA. The maps ϕ and γ define a weak equivalence between $\mathcal{M}[A, i]$ and $(H^*(\mathcal{M}[A, i]), 0)$, thereby showing that $\mathcal{M}[A, i]$ is formal. \square

Since $H^{\geq i+2}(\mathcal{M}[A, i]) = 0$, the equivalence of conditions (3) and (4) in the above lemma also follows from the (quite different) proof of Proposition 3.4 from [50]; see Remark 4.21 for more on this. We are now ready to prove descent for partial formality of DGAs.

Theorem 4.19. *Let (A^\bullet, d_A) be a DGA over \mathbb{k} , and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Suppose $H^{\leq i+1}(A)$ is finite-dimensional and $H^0(A) = \mathbb{k}$. Then (A^\bullet, d_A) is i -formal if and only if $(A^\bullet \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ is i -formal.*

Proof. Since $\dim H^{i+1}(\mathcal{M}(A, i)) \leq \dim H^{i+1}(A) < \infty$, we may apply Lemma 4.18 to infer that the DGA (A^\bullet, d_A) is i -formal if and only if the DGA $\mathcal{M}[A, i]$ is formal. By construction, $H^q(\mathcal{M}[A, i])$ equals $H^q(A)$ for $q \leq i$, injects into $H^q(A)$ for $q = i + 1$, and vanishes for $q > i + 1$. Hence, in view of our hypothesis, $H^*(\mathcal{M}[A, i])$ is of finite-type. By Corollary 4.17, $\mathcal{M}[A, i]$ is formal if and only if $\mathcal{M}[A, i] \otimes \mathbb{K}$ is formal. By Lemma 4.18 again, this is equivalent to the i -formality of $\mathcal{M}[A, i] \otimes \mathbb{K}$. \square

4.8. Formality notions for spaces. To every space X , Sullivan [81] associated in a functorial way a DGA of ‘rational polynomial forms’, denoted $A_{PL}^\bullet(X)$. As shown in [22, §10], there is a natural identification $H^*(A_{PL}^\bullet(X)) = H^*(X, \mathbb{Q})$ under which the respective induced homomorphisms in cohomology correspond. In particular, the weak isomorphism type of $A_{PL}^\bullet(X)$ depends only on the rational homotopy type of X .

A DGA (A, d) over \mathbb{k} is called a *model* for the space X if A is weakly equivalent to Sullivan’s algebra $A_{PL}(X; \mathbb{k}) := A_{PL}(X) \otimes_{\mathbb{Q}} \mathbb{k}$. In other words, $\mathcal{M}(A)$ is isomorphic to $\mathcal{M}(X; \mathbb{k}) := \mathcal{M}(X) \otimes_{\mathbb{Q}} \mathbb{k}$, where $\mathcal{M}(A)$ is the minimal model of A and $\mathcal{M}(X)$ is the minimal model of $A_{PL}(X)$. In the same vein, A is an i -model for X if $(A, d) \simeq_i A_{PL}(X; \mathbb{k})$. For instance, if X is a smooth manifold, then the de Rham algebra $\Omega_{dR}^\bullet(X)$ is a model for X over \mathbb{R} .

A space X is said to be *formal* over \mathbb{k} if the model $A_{PL}(X; \mathbb{k})$ is formal, that is, there is a quasi-isomorphism $\mathcal{M}(X; \mathbb{k}) \rightarrow (H^*(X; \mathbb{k}), d = 0)$. Likewise, X is said to be *i -formal*, for some $i \geq 0$, if there is an i -quasi-isomorphism $\mathcal{M}(A_{PL}(X; \mathbb{k}), i) \rightarrow (H^*(X; \mathbb{k}), d = 0)$. Note that X is 0-formal if and only if X is path-connected. Also, since a homotopy equivalence $X \simeq Y$ induces an isomorphism $H^*(Y; \mathbb{Q}) \xrightarrow{\cong} H^*(X; \mathbb{Q})$, it follows from Corollary 4.14 that the i -formality property is preserved under homotopy equivalences.

The following theorem of Papadima and Yuzvinsky [64] nicely relates the properties of the minimal model of X to the Koszulness of its cohomology algebra.

Theorem 4.20 ([64]). *Let X be a connected space with finite Betti numbers. If $\mathcal{M}(X) \cong \mathcal{M}(X, 1)$, then $H^*(X; \mathbb{Q})$ is a Koszul algebra. Moreover, if X is formal, then the converse holds.*

Remark 4.21. In [50, Prop. 3.4], Măcincic shows that every i -formal space X for which $H^{\geq i+2}(X; \mathbb{Q})$ vanishes is formal. In particular, the notions of formality and i -formality coincide for $(i + 1)$ -dimensional CW-complexes. In general, though, full formality is a much stronger condition than partial formality.

Remark 4.22. There is a competing notion of i -formality, due to Fernández and Muñoz [26]. As explained in [50], the two notions differ significantly, even for $i = 1$. In what follows, we will use exclusively the classical notion of i -formality given above.

As is well-known, the (full) formality property behaves well with respect to field extensions of the form $\mathbb{Q} \subset \mathbb{k}$. Indeed, it follows from Halperin and Stasheff’s Corollary 4.17 that a connected space X with finite Betti numbers is formal over \mathbb{Q} if and only if X is formal over \mathbb{k} . This result was first stated and proved by Sullivan [81], using different techniques. An independent proof was given by Neisendorfer and Miller [58] in the simply-connected case.

These classical results on descent of formality may be strengthened to a result on descent of partial formality. More precisely, using Theorem 4.19, we obtain the following immediate corollary.

Corollary 4.23. *Let X be a connected space with finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i -formal over \mathbb{Q} if and only if X is i -formal over \mathbb{k} .*

5. GROUPS, LIE ALGEBRAS, AND GRADED-FORMALITY

We now turn to finitely generated groups, and to two graded Lie algebras attached to each such group. We put special emphasis on the relationship between these Lie algebras, leading to the notion of graded-formality.

5.1. Central filtrations on groups. We start with some general background on lower central series and the associated graded Lie algebra of a group. For more details on this classical topic, we refer to Lazard [43] and Magnus et al. [51]. Let G be a group. For elements $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$ be their group commutator. Likewise, for subgroups $H, K < G$, let $[H, K]$ be the subgroup of G generated by all commutators $[x, y]$ with $x \in H, y \in K$.

A (central) filtration on the group G is a decreasing sequence of subgroups, $G = \mathcal{F}_1 G > \mathcal{F}_2 G > \mathcal{F}_3 G > \dots$, such that $[\mathcal{F}_r G, \mathcal{F}_s G] \subset \mathcal{F}_{r+s} G$. It is readily verified that, for each $k > 1$, the group $\mathcal{F}_{k+1} G$ is a normal subgroup of $\mathcal{F}_k G$, and the quotient group $\text{gr}_k^{\mathcal{F}}(G) = \mathcal{F}_k G / \mathcal{F}_{k+1} G$ is abelian. As before, let \mathbb{k} be a field of characteristic 0. The direct sum

$$(20) \quad \text{gr}^{\mathcal{F}}(G; \mathbb{k}) = \bigoplus_{k \geq 1} \text{gr}_k^{\mathcal{F}}(G) \otimes_{\mathbb{Z}} \mathbb{k}$$

is a graded Lie algebra over \mathbb{k} , with Lie bracket induced from the group commutator: If $x \in \mathcal{F}_r G$ and $y \in \mathcal{F}_s G$, then $[x + \mathcal{F}_{r+1} G, y + \mathcal{F}_{s+1} G] = xyx^{-1}y^{-1} + \mathcal{F}_{r+s+1} G$. We can view $\text{gr}^{\mathcal{F}}(-; \mathbb{k})$ as a functor from groups to graded \mathbb{k} -Lie algebras. Moreover, $\text{gr}^{\mathcal{F}}(G; \mathbb{K}) = \text{gr}^{\mathcal{F}}(G; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{K}$, for any field extension $\mathbb{k} \subset \mathbb{K}$. (Once again, if the underlying ring in a tensor product is understood, we will write \otimes for short.)

Let H be a normal subgroup of G , and let $Q = G/H$ be the quotient group. Define filtrations on H and Q by $\widetilde{\mathcal{F}}_k H = \mathcal{F}_k G \cap H$ and $\widetilde{\mathcal{F}}_k Q = \mathcal{F}_k G / \widetilde{\mathcal{F}}_k H$, respectively. We then have the following classical result of Lazard.

Proposition 5.1 (Thm. 2.4 in [43]). *The canonical projection $G \twoheadrightarrow G/H$ induces a natural isomorphism of graded Lie algebras, $\text{gr}^{\mathcal{F}}(G; \mathbb{k}) / \text{gr}^{\mathcal{F}}(H; \mathbb{k}) \xrightarrow{\cong} \text{gr}^{\mathcal{F}}(G/H; \mathbb{k})$.*

5.2. The associated graded Lie algebra. Any group G comes endowed with the lower central series (LCS) filtration $\{\Gamma_k G\}_{k \geq 1}$, defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [\Gamma_k G, G]$. If $\Gamma_k G \neq 1$ but $\Gamma_{k+1} G = 1$, then G is said to be a k -step nilpotent group. In general, though, the LCS filtration does not terminate.

The Lie algebra $\text{gr}(G; \mathbb{k}) = \text{gr}^{\Gamma}(G; \mathbb{k})$ is called the *associated graded Lie algebra* (over \mathbb{k}) of the group G . For instance, if $F = F_n$ is a free group of rank n , then $\text{gr}(F; \mathbb{k})$ is the free graded Lie algebra $\text{lie}(\mathbb{k}^n)$. A group homomorphism $f: G_1 \rightarrow G_2$ induces a morphism of graded Lie algebras, $\text{gr}(f; \mathbb{k}): \text{gr}(G_1; \mathbb{k}) \rightarrow \text{gr}(G_2; \mathbb{k})$; moreover, if f is surjective, then $\text{gr}(f; \mathbb{k})$ is also surjective.

For each $k \geq 2$, the factor group $G/\Gamma_k(G)$ is the maximal $(k-1)$ -step nilpotent quotient of G . The canonical projection $G \twoheadrightarrow G/\Gamma_k(G)$ induces an epimorphism $\text{gr}(G; \mathbb{k}) \twoheadrightarrow \text{gr}(G/\Gamma_k(G); \mathbb{k})$, which is an isomorphism in degrees $s < k$.

From now on, unless otherwise specified, we will assume that the group G is finitely generated. That is, there is a free group F of finite rank, and an epimorphism $\varphi: F \twoheadrightarrow G$. Let $R = \ker(\varphi)$; then $G = F/R$ is called a presentation for G . Note that the induced morphism $\text{gr}(\varphi; \mathbb{k}): \text{gr}(F; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$ is surjective. Thus, $\text{gr}(G; \mathbb{k})$ is a finitely generated Lie algebra, with generators in degree 1.

Let $H \triangleleft G$ be a normal subgroup, and let $Q = G/H$. Denote by $\widetilde{\Gamma}_r H = \Gamma_r G \cap H$ the induced filtration on H . It is readily seen that the filtration $\widetilde{\Gamma}_r Q = \Gamma_r G / \widetilde{\Gamma}_r H$ coincides with the LCS filtration on Q . Hence, by Proposition 5.1,

$$(21) \quad \text{gr}(Q; \mathbb{k}) \cong \text{gr}(G; \mathbb{k}) / \text{gr}^{\widetilde{\Gamma}}(H; \mathbb{k}).$$

Now suppose $G = H \rtimes Q$ is a semi-direct product of groups. In general, there is not much of a relation between the respective associated graded Lie algebras. Nevertheless, we have the following well-known result of Falk and Randell [21], which shows that $\text{gr}(G; \mathbb{k}) = \text{gr}(H; \mathbb{k}) \rtimes \text{gr}(Q; \mathbb{k})$ for ‘almost-direct’ products of groups.

Theorem 5.2 (Thm. 3.1 in [21]). *Let $G = H \rtimes Q$ be a semi-direct product of groups, and suppose Q acts trivially on H_{ab} . Then the filtrations $\{\widetilde{\Gamma}_r H\}_{r \geq 1}$ and $\{\Gamma_r H\}_{r \geq 1}$ coincide, and there is a split exact sequence of graded Lie algebras, $0 \rightarrow \text{gr}(H; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k}) \rightarrow \text{gr}(Q; \mathbb{k}) \rightarrow 0$.*

5.3. The holonomy Lie algebra. The holonomy Lie algebra of a finitely generated group was introduced by K.-T. Chen [12] and Kohno [39], and was further studied in [53, 59, 77].

Definition 5.3. Let G be a finitely generated group. The *holonomy Lie algebra* of G is the holonomy Lie algebra of the cohomology ring $A = H^*(G; \mathbb{k})$, that is,

$$(22) \quad \mathfrak{h}(G; \mathbb{k}) = \text{lie}(H_1(G; \mathbb{k})) / \langle \text{im}(\partial_G) \rangle,$$

where ∂_G is the dual to the cup-product map $\cup_G : H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \rightarrow H^2(G; \mathbb{k})$.

By construction, $\mathfrak{h}(G; \mathbb{k})$ is a quadratic Lie algebra. If $f : G_1 \rightarrow G_2$ is a group homomorphism, then the induced map in cohomology, $H^1(f) : H^1(G_2; \mathbb{k}) \rightarrow H^1(G_1; \mathbb{k})$, yields a morphism of graded Lie algebras, $\mathfrak{h}(f; \mathbb{k}) : \mathfrak{h}(G_1; \mathbb{k}) \rightarrow \mathfrak{h}(G_2; \mathbb{k})$. Moreover, if f is surjective, then $\mathfrak{h}(f; \mathbb{k})$ is also surjective. Finally, $\mathfrak{h}(G; \mathbb{K}) = \mathfrak{h}(G; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{K}$, for any field extension $\mathbb{k} \subset \mathbb{K}$.

In the definition of the holonomy Lie algebra of G , we used the cohomology ring of a classifying space $K(G, 1)$. As the next lemma shows, we may replace this space by any other connected CW-complex with the same fundamental group.

Lemma 5.4. *Let X be a connected CW-complex with $\pi_1(X) = G$. Then $\mathfrak{h}(H^*(X; \mathbb{k})) \cong \mathfrak{h}(G; \mathbb{k})$.*

Proof. We may construct a classifying space for G by adding cells of dimension 3 and higher to X in a suitable way. The inclusion map, $j : X \rightarrow K(G, 1)$, induces a map on cohomology rings, $H(j) : H^*(K(G, 1); \mathbb{k}) \rightarrow H^*(X; \mathbb{k})$, which is an isomorphism in degree 1 and an injection in degree 2. Consequently, $H^2(j)$ restricts to an isomorphism from $\text{im}(\cup_G)$ to $\text{im}(\cup_X)$. Taking duals, we find that $\text{im}(\partial_X) = \text{im}(\partial_G)$. The conclusion follows. \square

In particular, if K_G is the 2-complex associated to a presentation of G , then $\mathfrak{h}(G; \mathbb{k})$ is isomorphic to $\mathfrak{h}(H^*(K_G; \mathbb{k}))$. Let $\bar{\phi}_n(G) := \dim \mathfrak{h}_n(G; \mathbb{k})$ be the dimensions of the graded pieces of the holonomy Lie algebra of G . The next corollary is an algebraic version of the LCS formula from Papadima and Yuzvinsky [64], but with no formality assumption.

Corollary 5.5. *Let X be a connected CW-complex with $\pi_1(X) = G$, let $A = H^*(X; \mathbb{k})$ be its cohomology algebra, and let qA be the quadratic closure of A . Then $\prod_{n \geq 1} (1 - t^n)^{\bar{\phi}_n(G)} = \sum_{i \geq 0} b_{ii} t^i$, where $b_{ii} = \dim \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$. Moreover, if qA is a Koszul algebra, then $\prod_{n \geq 1} (1 - t^n)^{\bar{\phi}_n} = \text{Hilb}(\text{qA}, -t)$.*

Proof. The first claim follows from Lemma 5.4, the Poincaré–Birkhoff–Witt formula (8), and Löffwall’s formula from Proposition 3.6. The second claim follows from the Koszul duality formula stated in Corollary 3.8. \square

5.4. A comparison map. Once again, let G be a finitely generated group. The next lemma is known; for a proof and more details, we refer to [77].

Lemma 5.6 ([53, 59]). *There exists a natural epimorphism of graded \mathbb{k} -Lie algebras, $\Phi_G: \mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$, inducing isomorphisms in degrees 1 and 2. Furthermore, this epimorphism is natural with respect to field extensions $\mathbb{k} \subset \mathbb{K}$.*

Proposition 5.7. *Let $V = H_1(G; \mathbb{k})$. Suppose the associated graded Lie algebra $\mathfrak{g} = \text{gr}(G; \mathbb{k})$ has presentation $\text{lie}(V)/\mathfrak{r}$. Then the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$ has presentation $\text{lie}(V)/\langle \mathfrak{r}_2 \rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \text{lie}_2(V)$. Furthermore, if $A = U(\mathfrak{g})$, then $\mathfrak{h}(G; \mathbb{k}) = \mathfrak{h}((\mathfrak{q}A)^1)$.*

Proof. We have a natural exact sequence, which was first established in a particular case by Sullivan [80] and in general by Lambe [41],

$$(23) \quad 0 \longrightarrow (\Gamma_2 G / \Gamma_3 G \otimes \mathbb{k})^* \xrightarrow{\beta} H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \xrightarrow{\cup} H^2(G; \mathbb{k}),$$

where β is the dual of Lie bracket product. Taking the dual of (23), we find that $\text{im}(\partial_G) = \ker(\beta^*)$. Hence, $\langle \mathfrak{r}_2 \rangle = \langle \text{im}(\partial_G) \rangle$ as ideals of $\text{lie}(V)$; thus, $\mathfrak{h}(G; \mathbb{k}) = \text{lie}(V)/\langle \mathfrak{r}_2 \rangle$. The last claim follows from Corollary 3.5. \square

Recall that we denote by $\phi_n(G)$ and $\bar{\phi}_n(G)$ the dimensions on the n -th graded pieces of $\text{gr}(G; \mathbb{k})$ and $\mathfrak{h}(G; \mathbb{k})$, respectively. By Lemma 5.6, $\bar{\phi}_n(G) \geq \phi_n(G)$, for all $n \geq 1$, and equality always holds for $n \leq 2$. Nevertheless, these inequalities can be strict for $n \geq 3$.

5.5. Graded-formality. We continue our discussion of the associated graded and holonomy Lie algebras of a finitely generated group with a formality notion that will be important in the sequel.

Definition 5.8. A finitely generated group G is *graded-formal* (over \mathbb{k}) if the canonical projection $\Phi_G: \mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$ is an isomorphism of graded Lie algebras.

This notion was introduced by Lee in [45], where it is called graded 1-formality. Next, we give two alternate definitions, which oftentimes are easier to verify (see [77] for a proof).

Lemma 5.9. *A finitely generated group G is graded-formal over \mathbb{k} if and only if either*

- (1) $\text{gr}(G; \mathbb{k})$ is quadratic, or
- (2) $\dim_{\mathbb{k}} \mathfrak{h}_n(G; \mathbb{k}) = \dim_{\mathbb{k}} \text{gr}_n(G; \mathbb{k})$, for all $n \geq 1$.

The lemma implies that the definition of graded-formality is independent of the choice of coefficient field \mathbb{k} of characteristic 0. More precisely, we have the following corollary.

Corollary 5.10. *A finitely generated group G is graded-formal over \mathbb{k} if and only if it is graded-formal over \mathbb{Q} .*

Proof. The dimension of a finite-dimensional vector space does not change upon extending scalars. The conclusion follows at once from Lemma 5.9(2). \square

5.6. Split injections. We are now in a position to state and prove the main result of this section, which proves the first part of Theorem 1.3 from the Introduction.

Theorem 5.11. *Let G be a finitely generated group. Suppose there is a split monomorphism $\iota: K \rightarrow G$. If G is a graded-formal group, then K is also graded-formal.*

Proof. In view of our hypothesis, we have an epimorphism $\sigma: G \twoheadrightarrow K$ such that $\sigma \circ \iota = \text{id}$. In particular, K is also finitely generated. Furthermore, the induced maps $\mathfrak{h}(\iota)$ and $\text{gr}(\iota)$ are also injective. Let $\pi: F \twoheadrightarrow G$ be a presentation for G . There is then an induced presentation for K , given by the composition $\sigma \circ \pi: F \twoheadrightarrow K$. By Lemma 5.6, there exist epimorphisms Φ_1 and Φ making the following diagram commute:

$$(24) \quad \begin{array}{ccc} \mathfrak{h}(K; \mathbb{k}) & \xrightarrow{\Phi_1} & \text{gr}(K; \mathbb{k}) \\ \downarrow \mathfrak{h}(\iota) & & \downarrow \text{gr}(\iota) \\ \mathfrak{h}(G; \mathbb{k}) & \xrightarrow{\Phi} & \text{gr}(G; \mathbb{k}). \end{array}$$

If the group G is graded-formal, then Φ is an isomorphism of graded Lie algebras. Hence, the epimorphism Φ_1 is also injective, and so K is a graded-formal. \square

Theorem 5.12. *Let $G = K \rtimes Q$ be a semi-direct product of finitely generated groups. If G is graded-formal, then Q is graded-formal. If, moreover, Q acts trivially on K_{ab} , then K is also graded-formal.*

Proof. The first assertion follows at once from Theorem 5.11. So assume Q acts trivially on K_{ab} . By Theorem 5.2, there exists a split exact sequence of graded Lie algebras, which we record in the top row of the next diagram.

$$(25) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{gr}(K; \mathbb{k}) & \xrightleftharpoons{\quad} & \text{gr}(G; \mathbb{k}) & \xrightleftharpoons{\quad} & \text{gr}(Q; \mathbb{k}) \longrightarrow 0 \\ & & \uparrow \Phi_K & & \uparrow \Phi_G & & \uparrow \Phi_Q \\ & & \mathfrak{h}(K; \mathbb{k}) & \xrightarrow{\quad} & \mathfrak{h}(G; \mathbb{k}) & \xrightarrow{\quad} & \mathfrak{h}(Q; \mathbb{k}) \longrightarrow 0. \end{array}$$

Let $\iota: K \rightarrow G$ be the inclusion map. By the above, we have an epimorphism $\sigma: \text{gr}(G; \mathbb{k}) \twoheadrightarrow \text{gr}(K; \mathbb{k})$ such that $\sigma \circ \text{gr}(\iota) = \text{id}$, and so $\text{gr}(K; \mathbb{k})$ is finitely generated. By Proposition 5.7, the map σ induces a morphism $\bar{\sigma}: \mathfrak{h}(G; \mathbb{k}) \rightarrow \mathfrak{h}(K; \mathbb{k})$ such that $\bar{\sigma} \circ \mathfrak{h}(\iota) = \text{id}$. Consequently, $\mathfrak{h}(\iota)$ is injective. Therefore, the morphism $\Phi_K: \mathfrak{h}(K; \mathbb{k}) \rightarrow \text{gr}(K; \mathbb{k})$ is also injective, and so K is graded-formal. \square

If the second hypothesis of Theorem 5.12 does not hold, the subgroup K may not be graded-formal, even when G is 1-formal. This is illustrated by the following example, adapted from [60].

Example 5.13. Let $K = \langle x, y \mid [x, [x, y]], [y, [x, y]] \rangle$ be the discrete Heisenberg group. Consider the semidirect product $G = K \rtimes_{\phi} \mathbb{Z}$, defined by the automorphism $\phi: K \rightarrow K$ given by $x \rightarrow y, y \rightarrow xy$. We have that $b_1(G) = 1$, and so G is 1-formal, yet K is not graded-formal.

5.7. Products and coproducts. We conclude this section with a discussion of the functors gr and \mathfrak{h} and the notion of graded-formality behave with respect to products and coproducts.

Lemma 5.14 ([47, 61]). *The functors gr and \mathfrak{h} preserve products and coproducts, that is, we have the following natural isomorphisms of graded Lie algebras,*

$$\left\{ \begin{array}{l} \text{gr}(G_1 \times G_2; \mathbb{k}) \cong \text{gr}(G_1; \mathbb{k}) \times \text{gr}(G_2; \mathbb{k}) \\ \text{gr}(G_1 * G_2; \mathbb{k}) \cong \text{gr}(G_1; \mathbb{k}) * \text{gr}(G_2; \mathbb{k}), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \mathfrak{h}(G_1 \times G_2; \mathbb{k}) \cong \mathfrak{h}(G_1; \mathbb{k}) \times \mathfrak{h}(G_2; \mathbb{k}) \\ \mathfrak{h}(G_1 * G_2; \mathbb{k}) \cong \mathfrak{h}(G_1; \mathbb{k}) * \mathfrak{h}(G_2; \mathbb{k}). \end{array} \right.$$

Proof. The first statement on the $\text{gr}(-)$ functor is well-known, while the second statement is the main theorem from [47]. The statements regarding the $\mathfrak{h}(-)$ functor can be found in [61]. \square

Regarding graded-formality, we have the following result, which sharpens and generalizes Lemma 4.5 from Plantiko [65], and proves the first part of Theorem 1.4 from the Introduction.

Proposition 5.15. *Let G_1 and G_2 be two finitely generated groups. Then the following conditions are equivalent.*

- (1) G_1 and G_2 are graded-formal.
- (2) $G_1 * G_2$ is graded-formal.
- (3) $G_1 \times G_2$ is graded-formal.

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and coproduct $G_1 * G_2$, Theorem 5.11 shows that implications (2) \Rightarrow (1) and (3) \Rightarrow (1) hold. Implications (1) \Rightarrow (2) and (1) \Rightarrow (3) follow from Lemma 5.14 and the naturality of the map Φ from Definition 5.8. \square

6. MALCEV LIE ALGEBRAS AND FILTERED-FORMALITY

In this section we consider the Malcev Lie algebra of a finitely generated group, and study the ensuing notions of filtered-formality and 1-formality.

6.1. Pronilpotent completions and Malcev Lie algebras. Let G be a finitely generated group, and let $\{\Gamma_k G\}_{k \geq 1}$ be its LCS filtration. The successive quotients of G by these normal subgroups form a tower of finitely generated, nilpotent groups, $\cdots \rightarrow G/\Gamma_4 G \rightarrow G/\Gamma_3 G \rightarrow G/\Gamma_2 G = G_{\text{ab}}$.

Let \mathbb{k} be a field of characteristic 0. It is possible to replace each nilpotent quotient $N_k = G/\Gamma_k G$ by $N_k \otimes \mathbb{k}$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_k/\text{tors}(N_k)$ via a procedure which will be discussed in §9.1. The corresponding inverse limit, $\mathfrak{M}(G; \mathbb{k}) = \varprojlim_k ((G/\Gamma_k G) \otimes \mathbb{k})$, is a pronilpotent, filtered Lie group over \mathbb{k} , called the *pronilpotent completion*, or *Malcev completion* of G over \mathbb{k} . We denote by $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$ the canonical homomorphism from G to its completion.

Let $\mathfrak{Lie}((G/\Gamma_k G) \otimes \mathbb{k})$ be the Lie algebra of the nilpotent Lie group $(G/\Gamma_k G) \otimes \mathbb{k}$. The pronilpotent Lie algebra

$$(26) \quad \mathfrak{m}(G; \mathbb{k}) := \varprojlim_k \mathfrak{Lie}((G/\Gamma_k G) \otimes \mathbb{k}),$$

with the inverse limit filtration, is called the *Malcev Lie algebra* of G (over \mathbb{k}). By construction, $\mathfrak{m}(-; \mathbb{k})$ is a functor from the category of finitely generated groups to the category of complete, separated, filtered \mathbb{k} -Lie algebras.

6.2. Quillen's construction. A different approach was taken by Quillen in [70, App. A]. Let us briefly recall his construction. The group-algebra $\mathbb{k}G$ has a natural Hopf algebra structure, with comultiplication $\Delta: \mathbb{k}G \rightarrow \mathbb{k}G \otimes \mathbb{k}G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, and counit the augmentation map $\varepsilon: \mathbb{k}G \rightarrow \mathbb{k}$. Moreover, the set of *group-like* elements in this Hopf algebra, i.e., those elements x for which $\Delta(x) = x \otimes x$, gets identified with G under the canonical inclusion $G \hookrightarrow \mathbb{k}G$.

The powers of the augmentation ideal $I = \ker(\varepsilon)$ form a descending, multiplicative filtration of $\mathbb{k}G$. The I -adic completion of the group-algebra, $\widehat{\mathbb{k}G} = \varprojlim_k \mathbb{k}G/I^k$, comes equipped with a descending filtration, whose k -th term is $\widehat{I}^k = \varprojlim_{j \geq k} I^k/I^j$. Define the completed tensor product $\widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}$ as the completion of $\mathbb{k}G \otimes \mathbb{k}G$ with respect to the natural tensor product filtration. Applying the I -adic completion functor to the map Δ yields a comultiplication map $\widehat{\Delta}: \widehat{\mathbb{k}G} \rightarrow \widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}$, which makes $\widehat{\mathbb{k}G}$ into a complete Hopf algebra. It is then apparent that the canonical map to the completion, $\iota: \mathbb{k}G \rightarrow \widehat{\mathbb{k}G}$, is a morphism of filtered Hopf algebras.

An element $x \in \widehat{\mathbb{k}G}$ is called *primitive* if $\widehat{\Delta}x = x\widehat{\Delta}1 + 1\widehat{\Delta}x$. The set of all primitive elements in $\widehat{\mathbb{k}G}$, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a Lie algebra, isomorphic to the Malcev Lie algebra of G , that is, $\mathfrak{m}(G; \mathbb{k}) \cong \text{Prim}(\widehat{\mathbb{k}G})$.

The filtration topology on $\widehat{\mathbb{k}G}$ is a metric topology; hence, the filtration topology on $\mathfrak{m}(G; \mathbb{k})$ is also metrizable, and thus separated. We shall denote by $\text{gr}(\mathfrak{m}(G; \mathbb{k}))$ the associated graded Lie algebra of $\mathfrak{m}(G; \mathbb{k})$ with respect to the induced inverse limit filtration.

The set of all group-like elements in $\widehat{\mathbb{k}G}$, with multiplication inherited from $\widehat{\mathbb{k}G}$, forms a group, denoted $M(G; \mathbb{k})$. This group comes endowed with a complete, separated filtration, with terms $M(G; \mathbb{k}) \cap (1 + I^k)$. As shown by S. Jennings and Quillen, there is a filtration-preserving isomorphism $M(G; \mathbb{k}) \cong \mathfrak{M}(G; \mathbb{k})$, see Massuyeau [55] for details. Furthermore, there is a one-to-one, filtration-preserving correspondence between primitive elements and group-like elements via the exponential and logarithmic maps

$$(27) \quad \mathfrak{M}(G; \mathbb{k}) \subset 1 + \widehat{I} \begin{array}{c} \xleftarrow{\text{exp}} \\ \xrightarrow{\text{log}} \end{array} \widehat{I} \supset \mathfrak{m}(G; \mathbb{k}) .$$

Restricting the canonical map $\iota: \mathbb{k}G \rightarrow \widehat{\mathbb{k}G}$ to group-like elements, we obtain a homomorphism from G to its pronilpotent completion, $\kappa: G \rightarrow \mathfrak{M}(G; \mathbb{k})$. Composing this homomorphism with the logarithmic map, $\log: \mathfrak{M}(G; \mathbb{k}) \rightarrow \mathfrak{m}(G; \mathbb{k})$, we obtain a filtration-preserving map, $\rho: G \rightarrow \mathfrak{m}(G; \mathbb{k})$. As shown by Quillen in [69], the map ρ induces an isomorphism of graded Lie algebras,

$$(28) \quad \text{gr}(\rho): \text{gr}(G; \mathbb{k}) \xrightarrow{\cong} \text{gr}(\mathfrak{m}(G; \mathbb{k})) .$$

In particular, $\text{gr}(\mathfrak{m}(G; \mathbb{k}))$ is generated in degree 1.

6.3. Minimal models and Malcev Lie algebras. Every group G has a classifying space $K(G, 1)$, which can be chosen to be a connected CW-complex. Such a CW-complex is unique up to homotopy, and thus, up to rational homotopy equivalence. Hence, by the discussion from §4.8 the weak equivalence type of the Sullivan algebra $A = A_{PL}(K(G, 1))$ depends only on the isomorphism type of G . By Theorem 4.5, the DGA $A \otimes_{\mathbb{Q}} \mathbb{k}$ has a 1-minimal model, $\mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{k}, 1)$, unique up to isomorphism. Moreover, the assignment $G \rightsquigarrow \mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{k}, 1)$ is functorial.

Assume now that the group G is finitely generated. Let $\mathcal{M} = \mathcal{M}(G; \mathbb{k})$ be a 1-minimal model of G , with the canonical filtration constructed in (14). The starting point is the finite-dimensional vector space $\mathcal{M}_1^1 = V_1 := H^1(G; \mathbb{k})$. Each sub-DGA \mathcal{M}_i is a Hirsch extension of \mathcal{M}_{i-1} by the finite-dimensional vector space $V_i := \ker(H^2(\mathcal{M}_{i-1}) \rightarrow H^2(A))$. Define $\mathfrak{L}(G; \mathbb{k}) = \varprojlim_i \mathfrak{L}_i(G; \mathbb{k})$ as the pronilpotent Lie algebra associated to the 1-minimal model $\mathcal{M}(G; \mathbb{k})$ in the manner described in §4.2, and note that the assignment $G \rightsquigarrow \mathfrak{L}(G; \mathbb{k})$ is also functorial.

Theorem 6.1 ([9, 81]). *There exist a natural isomorphism between the towers of nilpotent Lie algebras $\{\mathfrak{L}_i(G; \mathbb{k})\}_i$ and $\{\mathfrak{m}(G/\Gamma_i G; \mathbb{k})\}_i$. Hence, there is a functorial isomorphism $\mathfrak{L}(G; \mathbb{k}) \cong \mathfrak{m}(G; \mathbb{k})$ of complete, filtered Lie algebras.*

For further details, we refer to [23, Thm. 7.5] and [29, Thm. 13.2]. The functorial isomorphism $\mathfrak{m}(G; \mathbb{k}) \cong \mathfrak{L}(G; \mathbb{k})$, together with the dualization correspondence $\mathfrak{L}(G; \mathbb{k}) \leftrightarrow \mathcal{M}(G; \mathbb{k})$ define adjoint functors between the category of Malcev Lie algebras of finitely generated groups and the category of 1-minimal models of finitely generated groups.

6.4. Filtered-formality of groups. We now define the notion of filtered-formality for groups (or, weak formality [45]), based on the notion of filtered-formality for Lie algebras from Definition 2.3.

Definition 6.2. A finitely generated group G is said to be *filtered-formal* (over \mathbb{k}) if its Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$, endowed with the inverse limit filtration from (26), is filtered-formal.

Here are some more direct ways to think of this notion.

Proposition 6.3. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if either*

- (1) $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{gr}}(G; \mathbb{k})$ as filtered Lie algebras, or
- (2) $\mathfrak{m}(G; \mathbb{k})$ admits a homogeneous presentation.

Proof. (1) The isomorphism (28) implies that $\mathrm{gr}(\mathfrak{m}(G; \mathbb{k})) \cong \mathrm{gr}(G; \mathbb{k})$. The forward implication follows straight from the definitions, while the backward implication follows from Lemma 2.4.

(2) Choose a presentation $\mathrm{gr}(G; \mathbb{k}) = \mathrm{lie}(H_1(G; \mathbb{k}))/r$, where r is a homogeneous ideal. Using Lemma 2.2, we obtain a homogeneous presentation for the Malcev Lie algebra of G , of the form $\mathfrak{m}(G; \mathbb{k}) = \widehat{\mathrm{lie}}(H_1(G; \mathbb{k}))/\bar{r}$. Conversely, if such a presentation exists, then $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{g}}$, where $\mathfrak{g} = \mathrm{lie}(H_1(G; \mathbb{k}))/r$. \square

The notion of filtered-formality can also be interpreted in terms of minimal models. Let $\mathcal{M}(G; \mathbb{k})$ be the 1-minimal model of G , endowed with the canonical filtration, which is the minimal DGA dual to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ under the correspondence described in §4.2. Likewise, let $\mathcal{N}(G; \mathbb{k})$ be the minimal DGA (generated in degree 1) corresponding to the pronipotent Lie algebra $\widehat{\mathfrak{gr}}(G; \mathbb{k})$. Recall that both $\mathcal{M}(G; \mathbb{k})$ and $\mathcal{N}(G; \mathbb{k})$ come equipped with increasing filtrations as in (14), which correspond to the inverse limit filtrations on $\mathfrak{m}(G; \mathbb{k})$ and $\widehat{\mathfrak{gr}}(G; \mathbb{k})$, respectively.

Proposition 6.4. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if either*

- (1) there is a filtration-preserving DGA isomorphism between $\mathcal{M}(G; \mathbb{k})$ and $\mathcal{N}(G; \mathbb{k})$, or
- (2) there is a DGA isomorphism $\mathcal{M}(G; \mathbb{k}) \cong \mathcal{N}(G; \mathbb{k})$ inducing the identity on first cohomology.

Proof. (1) Recall Proposition 6.3 that G is filtered-formal if and only if $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{gr}}(G; \mathbb{k})$, as filtered Lie algebras. Dualizing, this condition becomes equivalent to $\mathcal{M}(G; \mathbb{k}) \cong \mathcal{N}(G; \mathbb{k})$, as filtered DGAs.

(2) Recall that G is filtered-formal if and only if $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{gr}}(\mathfrak{m}(G; \mathbb{k}))$ inducing the identity on their associated graded Lie algebras. Likewise, both \mathcal{M}_1^1 and \mathcal{N}_1^1 can be canonically identified with $\mathrm{gr}_1(G; \mathbb{k})^* = H^1(G; \mathbb{k})$. The desired conclusion follows. \square

Here is another description of filtered-formality, suggested to us by R. Porter.

Theorem 6.5. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if the canonical 1-minimal model $\mathcal{M}(G; \mathbb{k})$ is filtered-isomorphic to a 1-minimal model \mathcal{M} with positive Hirsch weights.*

Proof. First suppose G is filtered-formal, and let $\mathcal{N} = \mathcal{N}(G; \mathbb{k})$ be the minimal DGA dual to $\mathfrak{Q} = \widehat{\mathfrak{gr}}(G; \mathbb{k})$. By Proposition 6.4, this DGA is a 1-minimal model for G . Since by construction $\mathfrak{Q} = \widehat{\mathfrak{gr}}(\mathfrak{Q})$, Lemma 4.6 shows that the differential on \mathcal{N} is homogeneous with respect to the Hirsch weights.

Now suppose \mathcal{M} is a 1-minimal model for G over \mathbb{k} , with homogeneous differential on Hirsch weights. By Lemma 4.6 again, the dual Lie algebra $\mathfrak{Q}(\mathcal{M})$ is filtered-formal. On the other hand, the assumption that $\mathcal{M} \cong \mathcal{M}(G; \mathbb{k})$ and Theorem 6.1 together imply that $\mathfrak{Q}(\mathcal{M}) \cong \mathfrak{m}(G; \mathbb{k})$. Hence, the group G is filtered-formal by Definition 6.2. \square

We conclude this section by showing that the definition of filtered-formality is independent of the choice of coefficient field \mathbb{k} of characteristic 0. We would like to thank Y. Cornulier for asking whether the next result holds, and for pointing out the connection it would have with [13, Thm. 3.14].

Proposition 6.6. *Let G be a finitely generated group, and let $\mathbb{Q} \subset \mathbb{k}$ be a field extension. Then G is filtered-formal over \mathbb{Q} if and only if G is filtered-formal over \mathbb{k} .*

Proof. Write $\mathfrak{m} = \mathfrak{m}(G; \mathbb{Q})$, and let $\mathfrak{g} = \text{gr}(G; \mathbb{Q})$, which we identify with $\text{gr}^{\widehat{\Gamma}}(\mathfrak{m})$. The claim follows from Theorem 2.7. \square

7. FILTERED-FORMALITY AND 1-FORMALITY

In this section, we consider the 1-formality property of finitely generated groups, and the way it relates to Massey products, graded-formality, and filtered-formality. We also study the way various formality properties behave under free and direct products, as well as retracts.

7.1. 1-formality of groups. Let \mathbb{k} be a field of characteristic 0. We start with a basic definition.

Definition 7.1. A finitely generated group G is called *1-formal* (over \mathbb{k}) if a classifying space $K(G, 1)$ is 1-formal over \mathbb{k} .

Since any two classifying spaces for G are homotopy equivalent, the discussion from §4.8 shows that this notion is well-defined. A similar argument shows that the 1-formality property of a path-connected space X depends only on its fundamental group, $G = \pi_1(X)$.

The next, well-known theorem provides an equivalent, purely group-theoretic definition of 1-formality. Although proofs can be found in the literature (see for instance Markl–Papadima [53], Carlson–Toledo [8], and Remark 7.3 below), we provide here an alternative proof, based on Theorem 4.10 and the discussion from §6.3.

Theorem 7.2. *A finitely generated group G is 1-formal over \mathbb{k} if and only if the Malcev Lie algebra of G is isomorphic to the degree completion of the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$.*

Proof. Let $\mathcal{M}(G; \mathbb{k}) = \mathcal{M}(A_{PL}(K(G, 1)), 1) \otimes_{\mathbb{Q}} \mathbb{k}$ be the 1-minimal model of G . The group G is 1-formal if and only if there exists a DGA morphism $\mathcal{M}(G; \mathbb{k}) \rightarrow (H^*(G; \mathbb{k}), d = 0)$ inducing an isomorphism in first cohomology and a monomorphism in second cohomology, i.e., $\mathcal{M}(G; \mathbb{k})$ is a 1-minimal model for $(H^*(G; \mathbb{k}), d = 0)$.

Let $\mathfrak{L}(G; \mathbb{k})$ be the dual Lie algebra of $\mathcal{M}(G; \mathbb{k})$. By Theorem 6.1, the Malcev Lie algebra of G is isomorphic to $\mathfrak{L}(G; \mathbb{k})$. By Theorem 4.10, the degree completion of the holonomy Lie algebra of G is isomorphic to $\mathfrak{L}(G; \mathbb{k})$. This completes the proof. \square

Remark 7.3. Theorem 7.2 admits the following generalization: if G is a finitely generated group, and if (A, d) is a connected DGA with $\dim A^1 < \infty$ whose 1-minimal model is isomorphic to $\mathcal{M}(G; \mathbb{k})$, then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is isomorphic to the completion with respect to the degree filtration of the Lie algebra $\mathfrak{h}(A, d) := \text{lie}((A^1)^*) / \langle \text{im}((d^1)^* + \mu_A^*) \rangle$. Proofs of this result are given in [2, 63]; related results can be found in [4, 5, 66].

An equivalent formulation of Theorem 7.2 is given by Papadima and Suciu in [60]: A finitely generated group G is 1-formal over \mathbb{k} if and only if its Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is isomorphic to the degree completion of a quadratic Lie algebra, as filtered Lie algebras. For instance, if $b_1(G)$ equals 0 or 1, then G is 1-formal.

Clearly, finitely generated free groups are 1-formal; indeed, if F is such a group, then $\mathfrak{m}(F; \mathbb{k}) \cong \widehat{\text{lie}}(H_1(F; \mathbb{k}))$. Other well-known examples of 1-formal groups include fundamental groups of compact Kähler manifolds [15], fundamental groups of complements of complex algebraic hypersurfaces [39], and the pure braid groups of surfaces of genus not equal to one [4, 31].

7.2. Massey products. A well-known obstruction to 1-formality is provided by the higher-order Massey products (introduced in [54]). For our purposes, we will discuss here only triple Massey products of degree 1 cohomology classes.

Let γ_1, γ_2 and γ_3 be cocycles of degrees 1 in the (singular) chain complex $C^*(G; \mathbb{k})$, with cohomology classes $u_i = [\gamma_i]$ satisfying $u_1 \cup u_2 = 0$ and $u_2 \cup u_3 = 0$. That is, we assume there are 1-cochains γ_{12} and γ_{23} such that $d\gamma_{12} = \gamma_1 \cup \gamma_2$ and $d\gamma_{23} = \gamma_2 \cup \gamma_3$. It is readily seen that the 2-cochain $\omega = \gamma_{12} \cup \gamma_3 + \gamma_1 \cup \gamma_{23}$ is, in fact, a cocycle. The set of all cohomology classes $[\omega]$ obtained in this way is the *Massey triple product* $\langle u_1, u_2, u_3 \rangle$. Due to the ambiguity in the choice of γ_{12} and γ_{23} , the Massey triple product $\langle u_1, u_2, u_3 \rangle$ is a representative of the coset

$$(29) \quad H^2(G; \mathbb{k}) / (u_1 \cup H^1(G; \mathbb{k}) + H^1(G; \mathbb{k}) \cup u_3).$$

Remark 7.4. In [67, Thm. 2], Porter gave a topological method for computing cup products and higher-order Massey products in $H^2(G; \mathbb{k})$. Building on work of Dwyer [18], Fenn and Sjerve [25] gave explicit formulas for Massey products in a commutator-relator group.

If a group G is 1-formal, then all triple Massey products vanish in the quotient \mathbb{k} -vector space from (29). However, if G is only graded-formal, these Massey products need not vanish. As we shall see in Example 7.5, even a one-relator group G may be graded-formal, yet not 1-formal.

Example 7.5. Let $G = \langle x_1, \dots, x_5 \mid [x_1, x_2][x_3, [x_4, x_5]] = 1 \rangle$. Using [77, Thm. 8.3], we see that the group G is graded-formal. On the other hand, by [25], the triple Massey product $\langle u_3, u_4, u_5 \rangle$ is non-zero (modulo indeterminacy). Thus, G is not 1-formal, and so G is not filtered-formal.

7.3. Filtered-formality, graded-formality and 1-formality. The next result pulls together the various formality notions for groups, and establishes the basic relationship among them.

Proposition 7.6. *A finitely generated group G is 1-formal if and only if G is graded-formal and filtered-formal.*

Proof. First suppose G is 1-formal. Then, by Theorem 7.2, $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{h}}(G; \mathbb{k})$, and thus, $\text{gr}(G; \mathbb{k}) \cong \mathfrak{h}(G; \mathbb{k})$, by (28). Hence, G is graded-formal, by Lemma 5.9(1). It follows that $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$, and hence G is filtered-formal, by Proposition 6.3.

Now suppose G filtered-formal. Then, by Proposition 6.3, we have that $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$. Thus, if G is also graded-formal, $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{h}}(G; \mathbb{k})$. Hence, G is 1-formal. \square

Using Propositions 6.6 and 7.6, and Corollary 5.10, we obtain the following corollary.

Corollary 7.7. *A finitely generated group G is 1-formal over \mathbb{Q} if and only if G is 1-formal over \mathbb{k} .*

In other words, the 1-formality property of a finitely generated group is independent of the choice of coefficient field of characteristic 0.

A filtered-formal group need not be 1-formal. Examples include some of the free nilpotent groups from Example 9.1 and the unipotent groups from Example 9.8. In fact, the triple Massey products in the cohomology of a filtered-formal group need not vanish (modulo indeterminacy).

Example 7.8. Let $G = F_2/\Gamma_3 F_2 = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = [x_2, [x_1, x_2]] = 1 \rangle$ be the Heisenberg group. Then G is filtered-formal, yet has non-trivial triple Massey products, $\langle u_1, u_1, u_2 \rangle$ and $\langle u_2, u_1, u_2 \rangle$, in $H^2(G; \mathbb{k})$. Hence, G is not graded-formal.

As shown by Hain in [31, 32] the Torelli groups in genus 4 or higher are 1-formal, but the Torelli group in genus 3 is filtered-formal, yet not graded-formal. The next two examples show that there are groups which are graded-formal but not filtered-formal.

Example 7.9. In [1], Bartholdi et al. consider two infinite families of groups corresponding to the Yang–Baxter equations. The first are the quasitriangular groups QTr_n , which have presentations with generators x_{ij} ($1 \leq i \neq j \leq n$), and relations $x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}$ and $x_{ij}x_{kl} = x_{kl}x_{ij}$ for distinct i, j, k, l . The second are the triangular groups Tr_n , each of which is the quotient of QTr_n by the relations of the form $x_{ij} = x_{ji}$ for $i \neq j$. As shown by Lee in [45], the groups QTr_n and Tr_n are all graded-formal. On the other hand, as indicated in [1], these groups are non-1-formal (and hence, not filtered-formal) for all $n \geq 4$. A detailed proof of this fact is given in [75, Cor. 6.7].

Example 7.10. Let G be the group with generators x_1, \dots, x_4 and relators $[x_2, x_3]$, $[x_1, x_4]$, and $[x_1, x_3][x_2, x_4]$. As noted in [77], G is graded-formal. On the other hand, using the Tangent Cone theorem of [16], one can show that the group G is not 1-formal. Therefore, G is not filtered-formal.

7.4. Examples from link theory. Let $L = (L_1, \dots, L_n)$ be an n -component link in S^3 . The link group, $G = \pi_1(X)$, is the fundamental group of the complement, $X = S^3 \setminus \bigcup_{i=1}^n L_i$. In general, a link group is not 1-formal. This phenomenon was first detected by W.S. Massey by means of his higher-order products [54], but the absence of graded-formality and especially filtered-formality can be even harder to detect.

Example 7.11. Let L be either the 2-component Whitehead link or the 3-component Borromean links. It follows from work of Hain [30] that the corresponding link groups G are not graded formal, and thus not 1-formal. The non-1-formality of these groups can also be detected by suitable higher-order Massey products. We refer to [74] for a complete answer to the formality question for complements of 2-component links.

Next, we give an example of a link group which is graded-formal, yet not filtered-formal.

Example 7.12. Let L be the link of 5 great circles in S^3 corresponding to the arrangement of transverse planes through the origin of \mathbb{R}^4 denoted as $\mathcal{A}(31425)$ in Matei–Suciu [56]. As noted in [77], work of Berceanu and Papadima [3] implies that the link group G is graded-formal. On the other hand, as noted in [16, Ex. 8.2], the Tangent Cone theorem does not hold for this group, and thus G is not 1-formal. Consequently, G is not filtered-formal.

7.5. Propagation of filtered-formality. The next theorem shows that filtered-formality is inherited upon taking nilpotent quotients. In §9, we will focus on exploring the formality properties of torsion-free nilpotent groups.

Theorem 7.13. *Let G be a finitely generated group, and suppose G is filtered-formal. Then all the nilpotent quotients $G/\Gamma_i(G)$ are filtered-formal.*

Proof. Set $\mathfrak{g} = \text{gr}(G; \mathbb{k})$ and $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$, and write $\mathfrak{g} = \bigoplus_{k \geq 1} \mathfrak{g}_k$. For each $i \geq 1$, the canonical projection $\phi_i: G \rightarrow G/\Gamma_i G$ induces an epimorphism of complete, filtered Lie algebras, $\mathfrak{m}(\phi_i): \mathfrak{m} \rightarrow \mathfrak{m}(G/\Gamma_i G; \mathbb{k})$. In each degree $k \geq i$, we have that $\widehat{\Gamma}_k \mathfrak{m}(G/\Gamma_i G; \mathbb{k}) = 0$, and so $\mathfrak{m}(\phi_i)(\widehat{\Gamma}_k \mathfrak{m}) = 0$. Therefore, there exists an induced epimorphism $\Phi_{k,i}: \mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m} \rightarrow \mathfrak{m}(G/\Gamma_i G; \mathbb{k})$.

Passing to the associated graded, we obtain an epimorphism $\text{gr}(\mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m}) \twoheadrightarrow \text{gr}(\mathfrak{m}(G/\Gamma_i G; \mathbb{k}))$, which is readily seen to be an isomorphism for $k = i$. Using now Lemma 2.1, we conclude that the map $\Phi_{i,i}$ is an isomorphism of (complete, separated) filtered Lie algebras.

On the other hand, our filtered-formality assumption on G allows us to identify $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{k \geq 1} \mathfrak{g}_k$. Let $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ be the canonical morphism. By construction, we have isomorphisms $\iota_k: \mathfrak{g}/\Gamma_k \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}/\widehat{\Gamma}_k \widehat{\mathfrak{g}}$ for all $k \geq 1$. Thus, $\mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m} \cong \widehat{\mathfrak{g}}/\widehat{\Gamma}_k \widehat{\mathfrak{g}} \cong \mathfrak{g}/\Gamma_k \mathfrak{g}$, for all $k \geq 1$. Using these identifications for $k = i$, together with the isomorphism $\Phi_{i,i}$ from above, we obtain isomorphisms $\mathfrak{m}(G/\Gamma_i G; \mathbb{k}) \cong \mathfrak{g}/\Gamma_i \mathfrak{g} \cong \text{gr}(G/\Gamma_i G; \mathbb{k})$, thereby showing that the nilpotent quotient $G/\Gamma_i G$ is filtered-formal. \square

Proposition 7.14. *Suppose $\phi: G_1 \rightarrow G_2$ is a homomorphism between two finitely generated groups, inducing an isomorphism $H_1(G_1; \mathbb{k}) \rightarrow H_1(G_2; \mathbb{k})$ and an epimorphism $H_2(G_1; \mathbb{k}) \rightarrow H_2(G_2; \mathbb{k})$.*

- (1) *If G_2 is 1-formal, then G_1 is also 1-formal.*
- (2) *If G_2 is filtered-formal, then G_1 is also filtered-formal.*
- (3) *If G_2 is graded-formal, then G_1 is also graded-formal.*

Proof. A celebrated theorem of Stallings [73] (see also [18, 27]) insures that the homomorphism ϕ induces isomorphisms $\phi_k: (G_1/\Gamma_k G_1) \otimes \mathbb{k} \rightarrow (G_2/\Gamma_k G_2) \otimes \mathbb{k}$, for all $k \geq 1$. Hence, ϕ induces an isomorphism $\mathfrak{m}(\phi): \mathfrak{m}(G_1; \mathbb{k}) \rightarrow \mathfrak{m}(G_2; \mathbb{k})$ between the respective Malcev completions, thereby proving claim (1). Using now the isomorphism (28), the other two claims follow at once. \square

7.6. Split injections. We are now ready to state and prove the main result of this section, which completes the proof of Theorem 1.3 from the Introduction.

Theorem 7.15. *Let G be a finitely generated group, and let $\iota: K \rightarrow G$ be a split injection.*

- (1) *If G is filtered-formal, then K is also filtered-formal.*
- (2) *If G is 1-formal, then K is also 1-formal.*

Proof. By hypothesis, we have an epimorphism $\sigma: G \twoheadrightarrow K$ such that $\sigma \circ \iota = \text{id}$. It follows that the induced maps $\mathfrak{m}(\iota)$ and $\widehat{\text{gr}}(\iota)$ are also split injections.

Let $\pi: F \twoheadrightarrow G$ be an epimorphism from a finitely generated free group F to G . Let $\pi_1 := \sigma \circ \pi: F \twoheadrightarrow K$; there is then a map $\iota_1: F \rightarrow F$ which is a lift of ι , that is, $\iota \circ \pi_1 = \pi \circ \iota_1$. Consider the following diagram (we suppress the coefficient field \mathbb{k} of characteristic zero from the notation).

$$(30) \quad \begin{array}{ccccccc} & & & J_1 & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \widehat{\text{gr}}(K) \\ & & \nearrow & \downarrow & & \downarrow & & \downarrow \widehat{\text{gr}}(\iota) \\ I_1 & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \mathfrak{m}(K) & & & \\ & & \downarrow & & \downarrow & & & \\ & & J & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \widehat{\text{gr}}(G) & \\ & & \downarrow \mathfrak{m}(\iota_1) & & \downarrow \text{id} & & \downarrow \mathfrak{m}(\iota) & \\ I & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \mathfrak{m}(G) & & & \end{array}$$

We have $\mathfrak{m}(\iota_1) = \widehat{\text{gr}}(\iota_1)$. By assumption, G is filtered-formal; hence, there exists a filtered Lie algebra isomorphism $\Phi: \mathfrak{m}(G) \rightarrow \widehat{\text{gr}}(G)$ as in diagram (30), which induces the identity on associated graded algebras. It follows that Φ is induced from the identity map of $\widehat{\text{lie}}(F)$ upon projecting onto source and target, i.e., the bottom right square in the diagram commutes.

First, we show that the identity map $\text{id}: \widehat{\text{lie}}(F) \rightarrow \widehat{\text{lie}}(F)$ in the above diagram induces an inclusion map $I_1 \rightarrow J_1$. Suppose there is an element $c \in \widehat{\text{lie}}(F)$ such that $c \in I_1$ and $c \notin J_1$, that is, $[c] = 0$ in

$\mathfrak{m}(K)$ and $[c] \neq 0$ in $\widehat{\text{gr}}(G)$. Since $\widehat{\text{gr}}(\iota)$ is injective, we have that $\widehat{\text{gr}}(\iota)([c]) \neq 0$, that is, $\widehat{\text{gr}}(\iota_1)(c) \notin I$. We also have $\mathfrak{m}(\iota)([c]) = 0 \in \mathfrak{m}(G)$, that is, $\mathfrak{m}(\iota_1)(c) \in J$. This contradicts the fact that the inclusion $I \hookrightarrow J$ is induced by the identity map. Thus, $I_1 \subset J_1$.

In view of the above, we may define a Lie algebra morphism $\Phi_1 : \mathfrak{m}(K) \rightarrow \widehat{\text{gr}}(K)$ as the quotient of the identity on $\widehat{\text{lie}}(F)$. By construction, Φ_1 is an epimorphism. We also have $\widehat{\text{gr}}(\iota) \circ \Phi_1 = \Phi \circ \mathfrak{m}(\iota)$. Since the maps $\mathfrak{m}(\iota)$, $\widehat{\text{gr}}(\iota)$ and Φ are all injective, the map Φ_1 is also injective. Therefore, Φ_1 is an isomorphism, and so the group K is filtered-formal.

Finally, part (2) follows at once from part (1) and Theorem 5.11. \square

This completes the proof of Theorem 1.3 from the Introduction. As we shall see in Example 7.9, this theorem is useful for deciding whether certain infinite families of groups are 1-formal.

We now proceed with the proof of Theorem 1.4. First, we need a lemma.

Lemma 7.16 ([16]). *Let G_1 and G_2 be two finitely generated groups. Then $\mathfrak{m}(G_1 \times G_2; \mathbb{k}) \cong \mathfrak{m}(G_1; \mathbb{k}) \times \mathfrak{m}(G_2; \mathbb{k})$ and $\mathfrak{m}(G_1 * G_2; \mathbb{k}) \cong \mathfrak{m}(G_1; \mathbb{k}) \hat{*} \mathfrak{m}(G_2; \mathbb{k})$.*

Proposition 7.17. *For finitely generated groups G_1 and G_2 , the following conditions are equivalent.*

- (1) G_1 and G_2 are filtered-formal.
- (2) $G_1 * G_2$ is filtered-formal.
- (3) $G_1 \times G_2$ is filtered-formal.

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and coproduct $G_1 * G_2$, we may apply Theorem 7.15 to conclude that implications (2) \Rightarrow (1) and (3) \Rightarrow (1) hold. Implications (1) \Rightarrow (2) and (1) \Rightarrow (3) follow from Lemmas 2.8, 2.9, and 7.16. \square

As we shall see in Example 7.12, the implication (1) \Rightarrow (3) from Proposition 7.17 cannot be strengthened from direct products to arbitrary semi-direct products. More precisely, there exist split extensions of the form $G = F_n \rtimes_{\alpha} \mathbb{Z}$, for certain automorphisms $\alpha \in \text{Aut}(F_n)$, such that the group G is not filtered-formal, although of course both F_n and \mathbb{Z} are 1-formal.

The next corollary follows at once from Propositions 5.15 and 7.17.

Corollary 7.18. *Suppose G_1 and G_2 are finitely generated groups such that G_1 is not graded-formal and G_2 is not filtered-formal. Then the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.*

As mentioned in § 1.2, concrete examples of groups which do not possess either formality property can be obtained by taking direct products of groups which enjoy one property but not the other.

8. DERIVED SERIES AND LIE ALGEBRAS

We now study some of the relationships between the derived series of a group and the derived series of the corresponding Lie algebras.

8.1. Derived series. Consider the derived series of a group G , starting at $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Note that any homomorphism $\phi : G \rightarrow H$ takes $G^{(i)}$ to $H^{(i)}$. The quotient groups, $G/G^{(i)}$, are solvable; in particular, $G/G' = G_{\text{ab}}$, while G/G'' is the maximal metabelian quotient of G .

Assume that G is a finitely generated group, and fix a coefficient field \mathbb{k} of characteristic 0.

Proposition 8.1 ([77]). *The holonomy Lie algebras $\mathfrak{h}(G/G^{(i)}; \mathbb{k})$ of the derived quotients of G are isomorphic to $\mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})'$ for $i = 1$, and are isomorphic to $\mathfrak{h}(G; \mathbb{k})$ for $i \geq 2$.*

The next theorem is the Lie algebra version of Theorem 3.5 from [59].

Theorem 8.2 ([59]). *For each $i \geq 2$, there is an isomorphism of complete, separated filtered Lie algebras, $\mathfrak{m}(G/G^{(i)}; \mathbb{k}) \cong \mathfrak{m}(G; \mathbb{k})/\mathfrak{m}(G; \mathbb{k})^{(i)}$, where $\mathfrak{m}(G; \mathbb{k})^{(i)}$ denotes the closure of $\mathfrak{m}(G; \mathbb{k})^{(i)}$ with respect to the filtration topology on $\mathfrak{m}(G; \mathbb{k})$.*

8.2. Chen Lie algebras. As before, let G be a finitely generated group. For each $i \geq 2$, the i -th Chen Lie algebra of G is defined to be the associated graded Lie algebra of the corresponding solvable quotient, $\text{gr}(G/G^{(i)}; \mathbb{k})$. Clearly, this construction is functorial.

The quotient map, $q_i: G \rightarrow G/G^{(i)}$, induces a surjective morphism $\text{gr}(q_i)$ between associated graded Lie algebras $\text{gr}_k(G; \mathbb{k})$ and $\text{gr}_k(G/G^{(i)}; \mathbb{k})$. Plainly, this morphism is the canonical identification in degree 1. In fact, the map $\text{gr}(q_i)$ is an isomorphism for each $k \leq 2^i - 1$, see [77].

We now specialize to the case when $i = 2$, originally studied by K.-T. Chen in [11]. The Chen ranks of G are defined as $\theta_k(G) := \dim_{\mathbb{k}}(\text{gr}_k(G/G''; \mathbb{k}))$. For free groups, Chen showed that $\theta_k(F_n) = (k-1) \binom{n+k-2}{k}$ for all $k \geq 2$. By analogy, let us define the holonomy Chen ranks of G as $\bar{\theta}_k(G) = \dim_{\mathbb{k}}(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(G; \mathbb{k})$. It is readily seen that $\bar{\theta}_k(G) \geq \theta_k(G)$, with equality for $k \leq 2$.

8.3. Chen Lie algebras and formality. We are now ready to state and prove the main result of this section, which (together with the first corollary following it) proves Theorem 1.5 from the Introduction.

Theorem 8.3. *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $q_i: G \rightarrow G/G^{(i)}$ induces a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$\Psi_G^{(i)}: \text{gr}(G; \mathbb{k})/\text{gr}(G; \mathbb{k})^{(i)} \longrightarrow \text{gr}(G/G^{(i)}; \mathbb{k}).$$

Moreover, if G is a filtered-formal group, then $\Psi_G^{(i)}$ is an isomorphism and the solvable quotient $G/G^{(i)}$ is filtered-formal.

Proof. The map $q_i: G \rightarrow G/G^{(i)}$ induces a natural epimorphism $\text{gr}(q_i)$ of graded \mathbb{k} -Lie algebras $\text{gr}(G; \mathbb{k})$ and $\text{gr}(G/G^{(i)}; \mathbb{k})$. By Proposition 5.1, this epimorphism factors through an isomorphism, $\text{gr}(G; \mathbb{k})/\widetilde{\text{gr}}(G^{(i)}; \mathbb{k}) \xrightarrow{\cong} \text{gr}(G/G^{(i)}; \mathbb{k})$, where $\widetilde{\text{gr}}$ denotes the graded Lie algebra associated with the filtration $\widetilde{\Gamma}_k G^{(i)} = \Gamma_k G \cap G^{(i)}$.

On the other hand, as shown by Labute in [40, Prop. 10], the Lie ideal $\text{gr}(G; \mathbb{k})^{(i)}$ is contained in $\widetilde{\text{gr}}(G^{(i)}; \mathbb{k})$. Therefore, the map $\text{gr}(q_i)$ factors through the claimed epimorphism $\Psi_G^{(i)}$, as indicated in the following commuting diagram,

$$(31) \quad \begin{array}{ccccc} \text{gr}(G; \mathbb{k}) & \longrightarrow & \text{gr}(G; \mathbb{k})/\text{gr}(G; \mathbb{k})^{(i)} & \longrightarrow & \text{gr}(G; \mathbb{k})/\widetilde{\text{gr}}(G^{(i)}; \mathbb{k}) \\ & \searrow \text{gr}(q_i) & \downarrow \Psi_G^{(i)} & \swarrow \cong & \\ & & \text{gr}(G/G^{(i)}; \mathbb{k}) & & \end{array}$$

Suppose now that G is filtered-formal, and set $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ and $\mathfrak{g} = \text{gr}(G; \mathbb{k})$. We may identify $\widehat{\mathfrak{g}} \cong \mathfrak{m}$. Let $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ be the inclusion into the completion. Passing to solvable quotients, we obtain a morphism of filtered Lie algebras, $\varphi^{(i)}: \mathfrak{g}/\mathfrak{g}^{(i)} \rightarrow \mathfrak{m}/\mathfrak{m}^{(i)}$. Passing to the associated graded Lie

algebras, we obtain the following commuting diagram:

$$(32) \quad \begin{array}{ccc} \mathfrak{g}/\mathfrak{g}^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \text{gr}(G/G^{(i)}; \mathbb{k}) \\ \downarrow \text{gr}(\varphi^{(i)}) & & \downarrow \cong \\ \text{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) & \xrightarrow{\cong} & \text{gr}(\mathfrak{m}(G/G^{(i)}; \mathbb{k})). \end{array}$$

All the graded Lie algebras in this diagram are generated in degree 1, and all the morphisms induce the identity in this degree. Therefore, the diagram is commutative. Moreover, the right vertical arrow from (32) is an isomorphism by Quillen's isomorphism (28), while the lower horizontal arrow is an isomorphism by Theorem 8.2.

Recall that, by assumption, $\mathfrak{m} = \widehat{\mathfrak{g}}$; therefore, the inclusion of filtered Lie algebras $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ induces a morphism between the following two exact sequences,

$$(33) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{\text{gr}}(\overline{\mathfrak{m}^{(i)}}) & \longrightarrow & \text{gr}(\mathfrak{m}) & \longrightarrow & \text{gr}(\mathfrak{m})/\widetilde{\text{gr}}(\overline{\mathfrak{m}^{(i)}}) \longrightarrow 0 \\ & & \uparrow \text{dotted} & & \uparrow \cong & & \uparrow \text{dotted} \\ 0 & \longrightarrow & \mathfrak{g}^{(i)} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}^{(i)} \longrightarrow 0. \end{array}$$

Here $\widetilde{\text{gr}}$ means taking the associated graded Lie algebra corresponding to the induced filtration. Using formulas (1) and (6), it can be shown that $\widetilde{\text{gr}}(\overline{\mathfrak{m}^{(i)}}) = \mathfrak{g}^{(i)}$. Therefore, the morphism $\mathfrak{g}/\mathfrak{g}^{(i)} \rightarrow \text{gr}(\mathfrak{m})/\widetilde{\text{gr}}(\overline{\mathfrak{m}^{(i)}})$ is an isomorphism. We also know that $\text{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) = \text{gr}(\mathfrak{m})/\widetilde{\text{gr}}(\overline{\mathfrak{m}^{(i)}})$. Hence, the map $\text{gr}(\varphi^{(i)})$ is an isomorphism, and so, by (32), the map $\Psi_G^{(i)}$ is an isomorphism, too.

By Lemma 2.1, the map $\varphi^{(i)}$ induces an isomorphism of complete, filtered Lie algebras between the degree completion of $\mathfrak{g}/\mathfrak{g}^{(i)}$ and $\overline{\mathfrak{m}^{(i)}}$. As shown above, $\Psi_G^{(i)}$ is an isomorphism; hence, its degree completion is also an isomorphism. Composing with the isomorphism from Theorem 8.2, we obtain an isomorphism between the degree completion $\widetilde{\text{gr}}(G/G^{(i)}; \mathbb{k})$ and the Malcev Lie algebra $\mathfrak{m}(G/G^{(i)}; \mathbb{k})$. This shows that the solvable quotient $G/G^{(i)}$ is filtered-formal. \square

Remark 8.4. As shown in [40, §3], the analogue of Theorem 8.3 does not hold if the ground field \mathbb{k} has characteristic $p > 0$. More precisely, there are pro- p groups G for which the morphisms $\Psi_G^{(i)}$ ($i \geq 2$) are not isomorphisms.

Returning now to the setup from Lemma 5.6, let us compose the canonical projection $\text{gr}(q_i): \text{gr}(G; \mathbb{k}) \rightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$ with the epimorphism $\Phi_G: \mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$. We obtain in this fashion an epimorphism $\mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$, which fits into the following commuting diagram (we will suppress the coefficient field \mathbb{k}):

$$(34) \quad \begin{array}{ccccc} & & \mathfrak{h}(G) & \xrightarrow{\Phi_G} & \text{gr}(G) \\ & \swarrow & & \searrow & \downarrow \\ \mathfrak{h}(G/G^{(i)}) & & & & \text{gr}(G/G^{(i)}) \\ & \searrow & & \swarrow & \downarrow \\ & & \mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} & \longrightarrow & \text{gr}(G)/\text{gr}(G)^{(i)}. \end{array}$$

Putting things together, we obtain the following corollary, which recasts Theorem 4.2 from [59] in a setting which is both functorial, and holds in wider generality. This corollary provides a way to detect non-1-formality of groups.

Corollary 8.5. *For each $i \geq 2$, there is a natural epimorphism of graded \mathbb{k} -Lie algebras, $\Phi_G^{(i)}: \mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$. Moreover, if G is 1-formal, then $\Phi_G^{(i)}$ is an isomorphism.*

Corollary 8.6. *Suppose the group G is 1-formal. Then, for each for $i \geq 2$, the solvable quotient $G/G^{(i)}$ is graded-formal if and only if $\mathfrak{h}(G; \mathbb{k})^{(i)}$ vanishes.*

Proof. By Proposition 8.1, the projection $q_i: G \rightarrow G/G^{(i)}$ induces an isomorphism $\mathfrak{h}(q_i): \mathfrak{h}(G; \mathbb{k}) \rightarrow \mathfrak{h}(G/G^{(i)}; \mathbb{k})$. Since G is 1-formal, Corollary 8.5 guarantees that the map $\Phi_G^{(i)}: \mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})^{(i)} \rightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$ is an isomorphism. The claim follows from the left square of diagram (34). \square

9. TORSION-FREE NILPOTENT GROUPS

In this section we study the graded-formality and filtered-formality properties of a well-known class of groups: that of finitely generated, torsion-free nilpotent groups. In the process, we prove Theorem 1.6 from the Introduction.

9.1. Nilpotent groups and Lie algebras. We start by reviewing the construction of the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G (see [10, 42, 52] for more details). There is a refinement of the upper central series of such a group, $G = G_1 > \cdots > G_n > G_{n+1} = 1$, with each subgroup G_i a normal subgroup of G_{i+1} , and each quotient G_i/G_{i+1} an infinite cyclic group. (The integer n is an invariant of the group, called the length of G .) Using this fact, we can choose a *Malcev basis* $\{u_1, \dots, u_n\}$ for G , which satisfies $G_i = \langle G_{i+1}, u_i \rangle$. Consequently, each element $u \in G$ can be written uniquely as $u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$.

Using this basis, the group G , as a set, can be identified with \mathbb{Z}^n via the map sending $u_1^{a_1} \cdots u_n^{a_n}$ to $a = (a_1, \dots, a_n)$. The multiplication in G takes the form $u_1^{a_1} \cdots u_n^{a_n} \cdot u_1^{b_1} \cdots u_n^{b_n} = u_1^{\rho_1(a,b)} \cdots u_n^{\rho_n(a,b)}$, where each map $\rho_i: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ is a rational polynomial function. This procedure identifies the group G with the group (\mathbb{Z}^n, ρ) , with multiplication the map $\rho = (\rho_1, \dots, \rho_n): \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$. Thus, we can define a simply-connected nilpotent Lie group $G \otimes \mathbb{k} = (\mathbb{k}^n, \rho)$ by extending the domain of ρ , which is called the *Malcev completion* of G .

The discrete group G is a subgroup of the real Lie group $G \otimes \mathbb{R}$. The quotient space, $M = (G \otimes \mathbb{R})/G$, is a compact manifold, called a *nilmanifold*. As shown in [52], the Lie algebra of $G \otimes \mathbb{R}$ is isomorphic to $\mathfrak{m}(G; \mathbb{R})$. It is readily apparent that the nilmanifold M is an Eilenberg–MacLane space of type $K(G, 1)$. As shown by K. Nomizu, the cohomology ring $H^*(M, \mathbb{R})$ is isomorphic to the cohomology ring of the Lie algebra $\mathfrak{m}(G; \mathbb{R})$.

The polynomial functions ρ_i have the form $\rho_i(a, b) = a_i + b_i + \tau_i(a_1, \dots, a_{i-1}, b_1, \dots, b_{i-1})$. Denote by $\sigma = (\sigma_1, \dots, \sigma_n)$ the quadratic part of ρ . Then \mathbb{k}^n can be given a Lie algebra structure, with bracket $[a, b] = \sigma(a, b) - \sigma(b, a)$. As shown in [42], this Lie algebra is isomorphic to $\mathfrak{m}(G; \mathbb{k})$.

The group (\mathbb{Z}^n, ρ) has canonical basis $\{e_i\}_{i=1}^n$, where e_i is the i -th standard basis vector. Then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = (\mathbb{k}^n, [,])$ has Lie bracket given by $[e_i, e_j] = \sum_{k=1}^n s_{i,j}^k e_k$, where $s_{i,j}^k = b_k(e_i, e_j) - b_k(e_j, e_i)$.

As is well-known, the Chevalley–Eilenberg complex $\wedge^*(\mathfrak{m}(G; \mathbb{k})^*)$ is a minimal model for $M = K(G, 1)$; see e.g. [24, Thm. 3.18]. Clearly, this model is generated in degree 1; thus, it is also a 1-minimal model for G . As shown in [34], the nilmanifold M is formal if and only if M is a torus.

9.2. Nilpotent groups and filtered-formality. Let G be a finitely generated, torsion-free nilpotent group, and let $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ be its Malcev Lie algebra, as described above. Note that $\text{gr}(\mathfrak{m}) = \mathbb{k}^n$ has the same basis e_1, \dots, e_n as \mathfrak{m} , but, as we shall see, the Lie bracket on $\text{gr}(\mathfrak{m})$ may be different. The

Lie algebra \mathfrak{m} (and thus, the group G) is filtered-formal if and only if $\mathfrak{m} \cong \widehat{\text{gr}}(\mathfrak{m}) = \text{gr}(\mathfrak{m})$, as filtered Lie algebras. In general, though, this isomorphism need not preserve the chosen basis.

Example 9.1. For any finitely generated free group F , the k -step, free nilpotent group $F/\Gamma_{k+1}F$ is filtered-formal. Indeed, F is 1-formal, and thus filtered-formal. Hence, by Theorem 7.13, each nilpotent quotient of F is also filtered-formal. In fact, as shown in [55, Cor. 2.14], $\mathfrak{m}(F/\Gamma_{k+1}F) \cong \mathbf{L}/(\Gamma_{k+1}\mathbf{L})$, where $\mathbf{L} = \text{lie}(F)$.

Example 9.2. Let G be the 3-step, rank 2 free nilpotent group F_2/Γ_4F_2 . Identifying G with \mathbb{Z}^5 as a set, then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = \mathbb{k}^5$ has Lie brackets given by $[e_1, e_2] = e_3 - e_4/2 - e_5$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, and $[e_i, e_j] = 0$, otherwise (see [42, 10]). It is readily checked that the identity map of \mathbb{k}^5 is not a Lie algebra isomorphism between $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ and $\text{gr}(\mathfrak{m})$. Moreover, the differential of the 1-minimal model $\mathcal{M}(G) = \wedge^*(\mathfrak{m}^*)$ is not homogeneous on the Hirsch weights, although \mathfrak{m} (and G) are filtered-formal.

Now consider a finite-dimensional, nilpotent Lie algebra \mathfrak{m} over a field \mathbb{k} of characteristic 0. It is readily seen that the filtered-formality of such a Lie algebra coincides with the notions of ‘Carnot’, ‘naturally graded’, ‘homogeneous’ and ‘quasi-cyclic’ which appear in [13, 14, 37, 38, 46].

The question whether the Carnot property descends from $\mathbb{k} = \mathbb{R}$ to \mathbb{Q} was first raised by Johnson in [37]. A positive answer was given in [14, Cor. 4.2], but, as pointed out by Cornulier in [13, Rem. 3.15], the proof of that result had a gap. The following proposition gives a complete solution to Johnson’s question.

Proposition 9.3 ([13]). *Let \mathfrak{m} be a finite-dimensional, nilpotent Lie algebra over a field \mathbb{k} of characteristic 0, and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Then \mathfrak{m} is Carnot over \mathbb{k} if and only if $\mathfrak{m} \otimes_{\mathbb{k}} \mathbb{K}$ is Carnot over \mathbb{K} .*

Our more general Theorem 2.7 allows us to recover Proposition 9.3 as an immediate corollary.

9.3. Torsion-free nilpotent groups and filtered-formality. We now study in more detail the filtered-formality properties of torsion-free nilpotent groups. We start by singling out a rather large class of groups which enjoy this property.

Theorem 9.4. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then G is filtered-formal.*

Proof. The lower central series of our group takes the form $G = \Gamma_1G > \Gamma_2G > \Gamma_3G = 1$. Let $\{x_1, \dots, x_n\}$ be a basis for $G/\Gamma_2G = \mathbb{Z}^n$, and let $\{y_1, \dots, y_m\}$ be a basis for $\Gamma_2G = \mathbb{Z}^m$. Then, as shown for instance by Igusa and Orr in [36, Lem. 6.1], the group G has presentation

$$(35) \quad G = \langle x_1, \dots, x_n, y_1, \dots, y_m \mid [x_i, x_j] = \prod_{k=1}^m y_k^{c_{i,j}^k}, [y_i, y_j] = 1, \text{ for } i < j; [x_i, y_j] = 1 \rangle.$$

Let $a, b \in \mathbb{Z}^{n+m}$. A routine computation shows that $\rho_i(a, b) = a_i + b_i$ for $1 \leq i \leq n$ and $\rho_{n+k}(a, b) = a_{n+k} + b_{n+k} - \sum_{j=1}^k \sum_{i=j+1}^n c_{j,i}^k a_i b_j$ for $1 \leq k \leq m$. Set $c_{j,i}^k = -c_{i,j}^k$ if $j > i$. It follows that the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = (\mathbb{k}^{n+m}, [,])$ has Lie bracket given on generators by $[e_i, e_j] = \sum_{k=1}^m c_{i,j}^k e_{n+k}$ for $1 \leq i \neq j \leq n$, and zero otherwise.

Turning now to the associated graded Lie algebra of our group, we have an additive decomposition, $\text{gr}(G; \mathbb{k}) = \text{gr}_1(G; \mathbb{k}) \oplus \text{gr}_2(G; \mathbb{k}) = \mathbb{k}^n \oplus \mathbb{k}^m$, where the first factor has basis $\{e_1, \dots, e_n\}$, the second factor has basis $\{e_{n+1}, \dots, e_{n+m}\}$, and the Lie bracket is given as above. Therefore, $\mathfrak{m}(G; \mathbb{k}) \cong \text{gr}(G; \mathbb{k})$, as filtered Lie algebras. Hence, G is filtered-formal. \square

It is known that all nilpotent Lie algebras of dimension 4 or less are filtered-formal, see for instance [13]. In general, though, finitely generated, torsion-free nilpotent groups need not be filtered-formal. We illustrate this phenomenon with two examples: the first one extracted from the work of Cornulier [13], and the second one adapted from the work of Lambe and Priddy [42]. In both examples, the nilpotent Lie algebra \mathfrak{m} in question may be realized as the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G .

Example 9.5. Let \mathfrak{m} be the 5-dimensional Lie algebra with non-zero Lie brackets given by $[e_1, e_3] = e_4$ and $[e_1, e_4] = [e_2, e_3] = e_5$. The center of \mathfrak{m} is 1-dimensional, generated by e_5 , while the center of $\text{gr}(\mathfrak{m})$ is 2-dimensional, generated by e_2 and e_5 . Therefore, $\mathfrak{m} \not\cong \text{gr}(\mathfrak{m})$, and so \mathfrak{m} is not filtered-formal. It follows that the nilpotent group G is not filtered-formal, either. Using Theorem 6.5, it is readily checked that the 1-minimal model $\mathcal{M}(G) = \wedge^*(\mathfrak{m}^*)$ does not have positive Hirsch weights; nevertheless, $\mathcal{M}(G)$ has positive weights, given by the index of the chosen basis.

Example 9.6. Let \mathfrak{m} be the 7-dimensional \mathbb{k} -Lie algebra with non-zero Lie brackets given on basis elements by $[e_2, e_3] = e_6$, $[e_2, e_4] = e_7$, $[e_2, e_5] = -e_7$, $[e_3, e_4] = e_7$, and $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Then $\text{gr}(\mathfrak{m})$ has the same additive basis as \mathfrak{m} , with non-zero brackets given by $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Plainly, $\text{gr}(\mathfrak{m})$ is metabelian, (i.e., its derived subalgebra is abelian), while \mathfrak{m} is not metabelian. Thus, once again, $\mathfrak{m} \not\cong \text{gr}(\mathfrak{m})$, and so both \mathfrak{m} and G are not filtered-formal. In this case, though, we cannot use the indexing of the basis to put positive weights on $\mathcal{M}(G)$.

9.4. Filtered-formality and Koszulness. Carlson and Toledo [8] classified finitely generated, 1-formal, nilpotent groups with first Betti number 5 or less, while Plantiko [65] gave sufficient conditions for the associated graded Lie algebras of such groups to be non-quadratic. The following proposition follows from Theorem 4.1 in [65] and Lemma 2.4 in [8].

Proposition 9.7 ([8, 65]). *Let G be a finitely generated, torsion-free, nilpotent group, and suppose there exists a non-zero decomposable element in the kernel of the cup product map $H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \rightarrow H^2(G; \mathbb{k})$. Then G is not graded-formal.*

Here an element $u \in H^2(G; \mathbb{k})$ is said to be decomposable if $u = v \wedge w$ for some $v, w \in H^1(G; \mathbb{k})$.

Example 9.8. Let $U_n(\mathbb{R})$ be the nilpotent Lie group of upper triangular matrices with 1's along the diagonal. The quotient $M = U_n(\mathbb{R})/U_n(\mathbb{Z})$ is a nilmanifold of dimension $N = n(n-1)/2$. The unipotent group $U_n(\mathbb{Z})$ has canonical basis $\{u_{ij} \mid 1 \leq i < j \leq n\}$, where u_{ij} is the matrix obtained from the identity matrix by putting 1 in position (i, j) . Moreover, $U_n(\mathbb{Z}) \cong (\mathbb{Z}^N, \rho)$, where $\rho_{ij}(a, b) = a_{ij} + b_{ij} + \sum_{i < k < j} a_{ik}b_{kj}$, see [42]. The unipotent group $U_n(\mathbb{Z})$ is filtered-formal; nevertheless, Proposition 9.7 shows that this group is not graded-formal for $n \geq 3$.

Proposition 9.9. *Let G be a finitely generated, torsion-free, nilpotent group, and suppose G is filtered-formal. Then G is abelian if and only if the algebra $U(\text{gr}(G; \mathbb{k}))$ is Koszul.*

Proof. We only need to prove the non-trivial direction. If the algebra $U = U(\text{gr}(G; \mathbb{k}))$ is Koszul, then the Lie algebra $\text{gr}(G; \mathbb{k})$ is quadratic, i.e., the group G is graded-formal. Under the assumption that G is filtered-formal, we then have that G is 1-formal.

Let M be the nilmanifold with fundamental group G . Then M is also 1-formal. By Nomizu's theorem, the cohomology ring $A = H^*(M; \mathbb{k})$ is isomorphic to the Yoneda algebra $\text{Ext}_U^*(\mathbb{k}, \mathbb{k})$. On the other hand, since U is Koszul, the Yoneda algebra is isomorphic to U^1 , which is also Koszul. Hence, A is a Koszul algebra. As shown in [64], if M is 1-formal and if A is Koszul, then M is formal. By [34], this happens if and only if M is a torus. This completes the proof. \square

Corollary 9.10. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then $U(\text{gr}(G; \mathbb{k}))$ is not Koszul.*

Example 9.11. Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_1, x_2][x_3, x_4] \rangle$. The group G is a 2-step, commutator-relators nilpotent group. Hence, by the above corollary, the enveloping algebra $U(\mathfrak{h}(G; \mathbb{k}))$ is not Koszul. In fact, $U(\mathfrak{h}(G; \mathbb{k}))^1$ is isomorphic to the quadratic algebra from Example 3.9, which is not Koszul.

10. SEIFERT FIBERED MANIFOLDS

We conclude with an analysis of the fundamental groups of orientable Seifert manifolds from a rational homotopy viewpoint.

10.1. Riemann surfaces and Seifert fibered spaces. To start with, let Σ_g be a closed, orientable surface of genus g . The fundamental group $\Pi_g = \pi_1(\Sigma_g)$ is a 1-relator group, with generators $x_1, y_1, \dots, x_g, y_g$ and a single relation, $[x_1, y_1] \cdots [x_g, y_g] = 1$. Since this group is trivial for $g = 0$, we will assume for now that $g > 0$. The cohomology algebra $A = H^*(\Sigma_g; \mathbb{k})$ is the quotient of the exterior algebra on generators $a_1, b_1, \dots, a_g, b_g$, in degree 1 by the ideal I generated by $a_i b_i - a_j b_j$, for $1 \leq i < j \leq g$, together with $a_i a_j, b_i b_j, a_i b_j, a_j b_i$, for $1 \leq i < j \leq g$. It is readily seen that the generators of I form a quadratic Gröbner basis for this ideal; therefore, A is a Koszul algebra.

The Riemann surface Σ_g is a compact Kähler manifold, and thus, a formal space. It follows from Theorem 4.20 that the minimal model of Σ_g is generated in degree one, i.e., $\mathcal{M}(\Sigma_g) = \mathcal{M}(\Sigma_g, 1)$. The formality of Σ_g also implies the 1-formality of Π_g . As a consequence, the associated graded Lie algebra $\text{gr}(\Pi_g; \mathbb{k})$ is isomorphic to the holonomy Lie algebra $\mathfrak{h}(\Pi_g; \mathbb{k})$. Using again the fact that A is a Koszul algebra, we deduce from Corollary 5.5 that $\prod_{k \geq 1} (1 - t^k)^{\phi_k(\Pi_g)} = 1 - 2gt + t^2$.

We will consider here only orientable, closed Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space an orientable surface of genus $g \geq 0$, and finitely many exceptional orbits, encoded in pairs of coprime integers $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$ with $\alpha_j \geq 2$. The obstruction to trivializing the bundle $\eta: M \rightarrow \Sigma_g$ outside tubular neighborhoods of the exceptional orbits is given by an integer $b = b(\eta)$. The group $\pi_\eta := \pi_1(M)$ has presentation with generators $x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h$ and relators $[x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = h^b$ and $z_i^{\alpha_i} h^{\beta_i} = 1$ ($i = 1, \dots, s$), where h is central.

For instance, if $s = 0$, the corresponding manifold, $M_{g,b}$, is the S^1 -bundle over Σ_g with Euler number b . Let $\pi_{g,b} := \pi_1(M_{g,b})$ be the fundamental group of this manifold. If $b = 0$, then $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, whereas if $b = 1$, then $\pi_{g,1} = \langle x_1, y_1, \dots, x_g, y_g, h \mid [x_1, y_1] \cdots [x_g, y_g] = h, h \text{ central} \rangle$. In particular, $M_{1,1}$ is the Heisenberg 3-dimensional nilmanifold and $\pi_{1,1}$ is the group from Example 7.8.

10.2. Minimal model. As shown in [71], the Euler number $e(\eta)$ of the Seifert bundle $\eta: M \rightarrow \Sigma_g$ satisfies $e(\eta) = -b(\eta) - \sum_{i=1}^s \beta_i / \alpha_i$. If $g = 0$, the group π_η has first Betti number 0 or 1, according to whether $e(\eta)$ is non-zero or 0. Thus, π_η is 1-formal, and the Malcev Lie algebra $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is either 0, or the completed free Lie algebra of rank 1. To analyze the case when $g > 0$, we will employ the minimal model of M , as constructed by Putinar in [68] (see also [23, §8.8]).

Theorem 10.1 ([68]). *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. The minimal model $\mathcal{M}(M)$ is the Hirsch extension $\mathcal{M}(\Sigma_g) \otimes_{\mathbb{k}} (\wedge(c), d)$, where the differential is given by $d(c) = 0$ if $e(\eta) = 0$, and $d(c) \in \mathcal{M}^2(\Sigma_g)$ represents a generator of $H^2(\Sigma_g; \mathbb{k})$ if $e(\eta) \neq 0$.*

More precisely, recall that Σ_g is formal, and so there is a quasi-isomorphism $f: \mathcal{M}(\Sigma_g) \rightarrow (H^*(\Sigma_g; \mathbb{k}), d = 0)$. Thus, there is an element $a \in \mathcal{M}^2(M)$ such that $d(a) = 0$ and $H(f)([a]) \neq 0$ in $H^2(\Sigma_g; \mathbb{k}) = \mathbb{k}$. We then set $d(c) = a$ in the second case.

To each Seifert fibration $\eta: M \rightarrow \Sigma_g$ as above, let us associate the S^1 -bundle $\bar{\eta}: M_{g, \epsilon(\eta)} \rightarrow \Sigma_g$, where $\epsilon(\eta) = 0$ if $e(\eta) = 0$, and $\epsilon(\eta) = 1$ if $e(\eta) \neq 0$. For instance, $M_{0,0} = S^2 \times S^1$ and $M_{0,1} = S^3$. The above theorem implies that $\mathcal{M}(M) \cong \mathcal{M}(M_{g, \epsilon(\eta)})$. Hence, we have the following corollary.

Corollary 10.2. *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space. The Malcev Lie algebra of the fundamental group $\pi_\eta = \pi_1(M)$ is given by $\mathfrak{m}(\pi_\eta; \mathbb{k}) \cong \mathfrak{m}(\pi_{g, \epsilon(\eta)}; \mathbb{k})$.*

Corollary 10.3. *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. Then M admits a minimal model with positive Hirsch weights.*

Proof. We know from §10.1 that the minimal model $\mathcal{M}(\Sigma_g)$ is formal, and generated in degree one (since $g > 0$). By Theorem 6.5, $\mathcal{M}(\Sigma_g)$ is isomorphic to a minimal model of Σ_g with positive Hirsch weights; denote this model by $\mathcal{H}(\Sigma_g)$. By Theorem 10.1 and Lemma 4.3, the Hirsch extension $\mathcal{H}(\Sigma_g) \otimes_{\mathbb{k}} \wedge(c)$ is a minimal model for M , generated in degree one. Moreover, the weight of c equals 1 if $e(\eta) = 0$, and equals 2 if $e(\eta) \neq 0$. Clearly, the differential d is homogeneous with respect to these weights, and this completes the proof. \square

Corollary 10.4. *The fundamental groups of orientable Seifert manifolds are filtered-formal.*

Proof. The claim follows at once from Theorem 6.5 and Corollary 10.3. Alternatively, the claim also follows from Theorem 10.6 and the definition of filtered-formality. \square

Using Theorem 10.1 and Lemma 4.3 again, we obtain a quadratic model for the Seifert manifold M in the case when the base has positive genus.

Corollary 10.5. *Suppose $g > 0$. Then M has a quadratic model of the form $(H^*(\Sigma_g; \mathbb{k}) \otimes \wedge(c), d)$, where $\deg(c) = 1$ and the differential d is given by $d(a_i) = d(b_i) = 0$ for $1 \leq i \leq g$, $d(c) = 0$ if $e(\eta) = 0$, and $d(c) = a_1 \wedge b_1$ if $e(\eta) \neq 0$.*

10.3. Malcev Lie algebra. We give now an explicit presentation for the Malcev Lie algebra of π_η as the degree completion of a certain graded Lie algebra.

Theorem 10.6. *The Malcev Lie algebra of π_η is the degree completion of the graded Lie algebra*

$$(36) \quad L(\pi_\eta) = \begin{cases} \langle \text{lie}(x_1, y_1, \dots, x_g, y_g, z) / \langle \sum_{i=1}^g [x_i, y_i] = 0, z \text{ central} \rangle & \text{if } e(\eta) = 0; \\ \langle \text{lie}(x_1, y_1, \dots, x_g, y_g, w) / \langle \sum_{i=1}^g [x_i, y_i] = w, w \text{ central} \rangle & \text{if } e(\eta) \neq 0, \end{cases}$$

where $\deg(w) = 2$ and the other generators have degree 1. Moreover, $\text{gr}(\pi_\eta; \mathbb{k}) \cong L(\pi_\eta)$.

Proof. The case $g = 0$ was already dealt with, so assume $g > 0$. There are two cases to consider.

If $e(\eta) = 0$, Corollary 10.2 says that $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is isomorphic to the Malcev Lie algebra of $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, which is a 1-formal group. Furthermore, we know that $\text{gr}(\Pi_g; \mathbb{k})$ is the quotient of $\text{lie}(2g)$ by the ideal generated by $\sum_{i=1}^g [x_i, y_i]$. Hence, $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is isomorphic to the degree completion of $\text{gr}(\Pi_g \times \mathbb{Z}) = \text{gr}(\Pi_g; \mathbb{k}) \times \text{gr}(\mathbb{Z}; \mathbb{k})$, which is precisely the Lie algebra $L(\pi_\eta)$ from (36).

If $e(\eta) \neq 0$, Corollary 10.5 provides a quadratic model for our Seifert manifold. Taking the Lie algebra dual to this quadratic model and using [5, Thm. 4.3.6] or [2, Thm. 3.1], we infer that $\mathfrak{m}(\pi_\eta)$ is isomorphic to the degree completion of the graded Lie algebra $L(\pi_\eta)$. Furthermore, by formula (28), there is an isomorphism $\text{gr}(\mathfrak{m}(\pi_\eta; \mathbb{k})) \cong \text{gr}(\pi_\eta; \mathbb{k})$. This completes the proof. \square

In follow-up work [77], we give a presentation for the holonomy Lie algebra of an orientable Seifert manifold group, and derive the following result.

Proposition 10.7. *If $g = 0$, the group π_η is always 1-formal, while if $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.*

Acknowledgments. We wish to thank Yves Cornulier, Ștefan Papadima, and Richard Porter for several useful comments regarding this work.

REFERENCES

1. Laurent Bartholdi, Benjamin Enriquez, Pavel Etingof, and Eric Rains, *Groups and Lie algebras corresponding to the Yang–Baxter equations*, *J. Algebra* **305** (2006), no. 2, 742–764.
2. Barbu Berceanu, Daniela Anca Măcinic, Ștefan Papadima, and Clement Radu Popescu, *On the geometry and topology of partial configuration spaces of Riemann surfaces*, *Algebr. Geom. Topol.* **17** (2017), no. 2, 1163–1188.
3. Barbu Berceanu and Ștefan Papadima, *Cohomologically generic 2-complexes and 3-dimensional Poincaré complexes*, *Math. Ann.* **298** (1994), no. 3, 457–480.
4. Roman Bezrukavnikov, *Koszul DG-algebras arising from configuration spaces*, *Geom. Funct. Anal.* **4** (1994), no. 2, 119–135.
5. Christin Bibby and Justin Hilburn, *Quadratic-linear duality and rational homotopy theory of chordal arrangements*, *Algebr. Geom. Topol.* **16** (2016), 2637–2661.
6. Richard Body, Mamoru Mimura, Hiroo Shiga, and Dennis Sullivan, *p-universal spaces and rational homotopy types*, *Comment. Math. Helv.* **73** (1998), no. 3, 427–442.
7. Damien Calaque, Benjamin Enriquez, and Pavel Etingof, *Universal KZB equations: the elliptic case*, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, *Progr. Math.*, vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 165–266.
8. James A. Carlson and Domingo Toledo, *Quadratic presentations and nilpotent Kähler groups*, *J. Geom. Anal.* **5** (1995), no. 3, 359–377.
9. Bohumil Cenkl and Richard Porter, *Mal'cev's completion of a group and differential forms*, *J. Differential Geom.* **15** (1980), no. 4, 531–542 (1981).
10. Bohumil Cenkl and Richard Porter, *Nilmanifolds and associated Lie algebras over the integers*, *Pacific J. Math.* **193** (2000), no. 1, 5–29.
11. Kuo-Tsai Chen, *Integration in free groups*, *Ann. of Math. (2)* **54** (1951), 147–162.
12. Kuo-Tsai Chen, *Iterated integrals of differential forms and loop space homology*, *Ann. of Math. (2)* **97** (1973), 217–246.
13. Yves Cornulier, *Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups*, *Bull. Math. Soc. France* **14** (2016), no. 4, 693–744.
14. Karel Dekimpe and Kyung Bai Lee, *Expanding maps on infra-nilmanifolds of homogeneous type*, *Trans. Amer. Math. Soc.* **355** (2003), no. 3, 1067–1077.
15. Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan, *Real homotopy theory of Kähler manifolds*, *Invent. Math.* **29** (1975), no. 3, 245–274.
16. Alexandru Dimca, Ștefan Papadima, and Alexander I. Suciuc, *Topology and geometry of cohomology jump loci*, *Duke Math. J.* **148** (2009), no. 3, 405–457.
17. Vladimir Drinfel'd, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$* , *Leningrad Math. J.* **2** (1991), no. 4, 829–860.
18. William G. Dwyer, *Homology, Massey products and maps between groups*, *J. Pure Appl. Algebra* **6** (1975), no. 2, 177–190.
19. Torsten Ekedahl and Sergei Merkulov, *Grothendieck–Teichmüller group in algebra, geometry and quantization: A survey*, preprint (2011).
20. Benjamin Enriquez, *Elliptic associators*, *Selecta Math. (N.S.)* **20** (2014), no. 2, 491–584.
21. Michael Falk and Richard Randell, *The lower central series of a fiber-type arrangement*, *Invent. Math.* **82** (1985), no. 1, 77–88.

22. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.
23. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory II*, World Scientific Publishing, Hackensack, NJ, 2015.
24. Yves Félix, John Oprea, and Daniel Tanré, *Algebraic models in geometry*, Oxford Grad. Texts in Math., vol. 17, Oxford Univ. Press, Oxford, 2008.
25. Roger Fenn and Denis Sjerve, *Massey products and lower central series of free groups*, *Canad. J. Math.* **39** (1987), no. 2, 322–337.
26. Marisa Fernández and Vicente Muñoz, *Formality of Donaldson submanifolds*, *Math. Z.* **250** (2005), no. 1, 149–175.
27. Michael Freedman, Richard M. Hain, and Peter Teichner, *Betti number estimates for nilpotent groups*, Fields Medalists' lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 413–434.
28. Ralph Fröberg, *Koszul algebras*, Advances in commutative ring theory (Fez, 1997), Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999, pp. 337–350.
29. Phillip Griffiths and John Morgan, *Rational homotopy theory and differential forms*, Second ed., Progr. Math., vol. 16, Springer, New York, 2013.
30. Richard M. Hain, *Iterated integrals, intersection theory and link groups*, *Topology* **24** (1985), no. 1, 45–66.
31. Richard M. Hain, *Infinitesimal presentations of the Torelli groups*, *J. Amer. Math. Soc.* **10** (1997), no. 3, 597–651.
32. Richard M. Hain, *Genus 3 mapping class groups are not Kähler*, *J. Topol.* **8** (2015), no. 1, 213–246.
33. Stephen Halperin and James Stasheff, *Obstructions to homotopy equivalences*, *Adv. in Math.* **32** (1979), no. 3, 233–279.
34. Keizo Hasegawa, *Minimal models of nilmanifolds*, *Proc. Amer. Math. Soc.* **106** (1989), no. 1, 65–71.
35. Peter J. Hilton and Urs Stammach, *A course in homological algebra*, Second ed., Grad. Texts in Math., vol. 4, Springer-Verlag, New York, 1997.
36. Kiyoshi Igusa and Kent E. Orr, *Links, pictures and the homology of nilpotent groups*, *Topology* **40** (2001), no. 6, 1125–1166.
37. R. Warren Johnson, *Homogeneous Lie algebras and expanding automorphisms*, *Proc. Amer. Math. Soc.* **48** (1975), no. 2, 292–296.
38. Hisashi Kasuya, *Singularity of the varieties of representations of lattices in solvable Lie groups*, *J. Topol. Anal.* **8** (2016), no. 2.
39. Toshitake Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, *Nagoya Math. J.* **92** (1983), 21–37.
40. John P. Labute, *Fabulous pro-p-groups*, *Ann. Sci. Math. Québec* **32** (2008), no. 2, 189–197.
41. Larry A. Lambe, *Two exact sequences in rational homotopy theory relating cup products and commutators*, *Proc. Amer. Math. Soc.* **96** (1986), no. 2, 360–364.
42. Larry A. Lambe and Stewart B. Priddy, *Cohomology of nilmanifolds and torsion-free, nilpotent groups*, *Trans. Amer. Math. Soc.* **273** (1982), no. 1, 39–55.
43. Michel Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, *Ann. Sci. École Norm. Sup. (3)* **71** (1954), 101–190.
44. Andrey Lazarev and Martin Markl, *Disconnected rational homotopy theory*, *Adv. Math.* **283** (2015), 303–361.
45. Peter Lee, *The pure virtual braid group is quadratic*, *Selecta Math. (N.S.)* **19** (2013), no. 2, 461–508.
46. George Leger, *Derivations of Lie algebras. III*, *Duke Math. J.* **30** (1963), 637–645.
47. Alexander I. Lichtman, *On Lie algebras of free products of groups*, *J. Pure Appl. Algebra* **18** (1980), no. 1, 67–74.
48. Clas Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Math., vol. 1183, Springer, Berlin, 1986, pp. 291–338.
49. Mohamad Maassarani, *Sur certains espaces de configurations associés aux sous-groupes finis de $\mathrm{PSL}_2(\mathbb{C})$* , preprint (2015), [arXiv:1510.00617v1](https://arxiv.org/abs/1510.00617v1).
50. Anca Daniela Măcinic, *Cohomology rings and formality properties of nilpotent groups*, *J. Pure Appl. Algebra* **214** (2010), no. 10, 1818–1826.
51. Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers, New York-London-Sydney, 1966.
52. Anatoli I. Malcev, *On a class of homogeneous spaces*, *Amer. Math. Soc. Translation* **1951** (1951), no. 39, 33pp.
53. Martin Markl and Ștefan Papadima, *Homotopy Lie algebras and fundamental groups via deformation theory*, *Ann. Inst. Fourier (Grenoble)* **42** (1992), no. 4, 905–935.

54. William S. Massey, *Higher order linking numbers*, J. Knot Theory Ramifications **7** (1998), no. 3, 393–414.
55. Gwénaél Massuyeau, *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*, Bull. Soc. Math. France **140** (2012), no. 1, 101–161.
56. Daniel Matei and Alexander I. Suciú, *Homotopy types of complements of 2-arrangements in \mathbf{R}^4* , Topology **39** (2000), no. 1, 61–88.
57. John W. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 137–204.
58. Joseph Neisendorfer and Timothy Miller, *Formal and coformal spaces*, Illinois J. Math. **22** (1978), no. 4, 565–580.
59. Stefan Papadima and Alexander I. Suciú, *Chen Lie algebras*, Int. Math. Res. Not. (2004), no. 21, 1057–1086.
60. Stefan Papadima and Alexandru Suciú, *Geometric and algebraic aspects of 1-formality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52(100)** (2009), no. 3, 355–375.
61. Ștefan Papadima and Alexander I. Suciú, *Non-abelian resonance: product and coproduct formulas*, Bridging algebra, geometry, and topology, Springer Proc. Math. Stat., vol. 96, Springer, Cham, 2014, pp. 269–280.
62. Stefan Papadima and Alexander I. Suciú, *The topology of compact Lie group actions through the lens of finite models*, to appear in International Mathematics Research Notices, doi:10.1093/imrn/rnx294.
63. Stefan Papadima and Alexander I. Suciú, *Infinitesimal finiteness obstructions*, J. London Math. Soc. **99** (2019), no. 1, 173–193.
64. Stefan Papadima and Sergey Yuzvinsky, *On rational $K[\pi, 1]$ spaces and Koszul algebras*, J. Pure Appl. Algebra **144** (1999), no. 2, 157–167.
65. Rüdiger Plantiko, *The graded Lie algebra of a Kähler group*, Forum Math. **8** (1996), no. 5, 569–583.
66. Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005.
67. Richard Porter, *Milnor’s $\bar{\mu}$ -invariants and Massey products*, Trans. Amer. Math. Soc. **257** (1980), no. 1, 39–71.
68. Gabriela Putinar, *Minimal models and the virtual degree of Seifert fibered spaces*, Matematiche (Catania) **53** (1998), no. 2, 319–329 (1999).
69. Daniel G. Quillen, *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418.
70. Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.
71. Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.
72. Jean-Pierre Serre, *Lie algebras and Lie groups*, second ed., Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 1992, 1964 lectures given at Harvard University.
73. John Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
74. Alexander I. Suciú, *Cohomology jump loci of 3-manifolds*, preprint (2019), arXiv:1901.01419v1.
75. Alexander I. Suciú and He Wang, *Pure virtual braids, resonance, and formality*, Math. Zeit. **286** (2017), no. 3–4, 1495–1524.
76. Alexander I. Suciú and He Wang, *The pure braid groups and their relatives*, in: *Perspectives in Lie theory*, 403–426, Springer INdAM series, vol. 19, Springer, Cham, 2017.
77. Alexander I. Suciú and He Wang, *Cup products, lower central series, and holonomy Lie algebras*, J. Pure. Appl. Algebra. **223** (2019), no. 8, 3359–3385.
78. Alexander I. Suciú and He Wang, *Chen ranks and resonance varieties of the upper McCool groups*, preprint (2018), arXiv:1804.06006v1.
79. Alexander I. Suciú and He Wang, *Taylor expansions of groups and filtered-formality*, preprint (2019).
80. Dennis Sullivan, *On the intersection ring of compact three manifolds*, Topology **14** (1975), no. 3, 275–277.
81. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331.

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA

E-mail address: a.suciu@northeastern.edu

URL: <http://web.northeastern.edu/suciu/>

DEPARTMENT OF MATHEMATICS AND STATISTICS MS0084, UNIVERSITY OF NEVADA, RENO, NV 89557, USA

E-mail address: wanghemath@gmail.com, hew@unr.edu

URL: <http://wolfweb.unr.edu/homepage/hew/>