

ABELIAN DUALITY AND PROPAGATION OF RESONANCE

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ABSTRACT. We explore the relationship between a certain “abelian duality” property of spaces and the propagation properties of their cohomology jump loci. To that end, we develop the analogy between abelian duality spaces and those spaces which possess what we call the “EPY property.” The same underlying homological algebra allows us to deduce the propagation of jump loci: in the former case, characteristic varieties propagate, and in the latter, the resonance varieties. We apply the general theory to arrangements of linear and elliptic hyperplanes, as well as toric complexes, right-angled Artin groups, and Bestvina–Brady groups. Our approach brings to the fore the relevance of the Cohen–Macaulay condition in this combinatorial context.

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1. INTRODUCTION

It has long been recognized that Poincaré duality imposes constraints not just on the integral cohomology of a closed, orientable manifold, but also on the cohomology with twisted coefficients of such a manifold. Our goal here is to explore the constraints imposed by a different type of duality property, one which applies to other types of spaces, such as complements of hyperplane arrangements and certain polyhedral products.

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1.1. Duality and abelian duality. Duality groups were introduced by Bieri and Eckmann [BE73]: they are characterized by a cohomological vanishing condition less restrictive than Poincaré duality, and they possess a more general isomorphism between homology and cohomology of group representations. In this paper, we introduce an independent but related notion which we call abelian duality and which, we see, explains some previously conjectural behavior of the cohomology of abelian representations.

More precisely, let X be a connected, finite-type CW complex, with fundamental group G . Following Bieri and Eckmann [BE73], we say that X is a *duality space* of dimension n if $H^q(X, \mathbb{Z}G) = 0$ for $q \neq n$ and $H^n(X, \mathbb{Z}G)$ is non-zero and torsion-free. This type of duality property bears a formal resemblance to Serre duality for Cohen–Macaulay schemes, which itself can be understood in terms of Verdier duality. In section §4.2, we review Bieri and Eckmann’s definition in a slightly more general algebraic context in order to note such parallels with other homological duality theories.

By analogy, we say that a space X is an *abelian duality space* of dimension n if the analogous condition, with $\mathbb{Z}G$ replaced by $\mathbb{Z}G^{\text{ab}}$ is satisfied. In that case, we call the module $D := H^n(X, \mathbb{Z}G^{\text{ab}})$ the (abelian) dualizing module. Just as in the classical case, the abelian duality property gives rise to duality isomorphisms of the form

$$H^i(X, A) \cong H_{n-i}(G^{\text{ab}}, D \otimes_{\mathbb{Z}} A),$$

for any abelian representation A of G and for all i (Proposition 4.8). By restricting to abelian representations, however, the commutativity of the base ring $\mathbb{Z}G^{\text{ab}}$ imposes additional structure and reveals some interesting geometry. We give examples that show that the two properties are independent. Examples of duality groups which are not abelian duality groups are abundant, while examples with the latter but not the former property can be contrived.

Furthermore, we examine the behavior of abelian duality under fibrations. We say that a fibration sequence $F \rightarrow E \rightarrow B$ is *ab-exact* if $\pi_1(B)$ acts trivially on $H_1(F, \mathbb{Z})$, and the transgression map $H_2(B, \mathbb{Z}) \rightarrow H_1(F, \mathbb{Z})$ is zero. With this (essential) hypothesis and a finite-type condition, we show that, if two out of three of the spaces in the fibration sequence are abelian duality spaces, then so is the third (Proposition 4.13).

1.2. Propagation of characteristic varieties. Now let \mathbb{k} be an algebraically closed field, and regard $\mathbb{k}G^{\text{ab}}$ as the coordinate ring of the one-dimensional representation variety of G , that is, the character group $\widehat{G} := \text{Hom}(G, \mathbb{k}^*)$. Computing homology with coefficients in the rank 1 local systems parametrized by \widehat{G} yields the (depth 1) *characteristic varieties* $\mathcal{V}^p(X, \mathbb{k})$, consisting of those characters ρ for which $H_p(X, \mathbb{k}_\rho) \neq 0$. The geometry of these varieties controls to a large extent the homology of regular, abelian covers of X , see for instance [DS14, Su14, Su14'] and references therein.

Since X is assumed to be connected, $\mathcal{V}^0(X, \mathbb{k})$ has just one point, the trivial character $\mathbb{1}$. The higher-degree characteristic varieties, though, may be arbitrarily complicated. Nevertheless, in our situation, a nice pattern holds.

Theorem 1.1. *Suppose X is an abelian duality space of dimension n over \mathbb{k} . Then the characteristic varieties of X propagate: that is, for any character $\rho \in \widehat{G}$, if $H^p(X, \mathbb{k}_\rho) \neq 0$, then $H^q(X, \mathbb{k}_\rho) \neq 0$ for all $p \leq q \leq n$. Equivalently,*

$$\{\mathbb{1}\} = \mathcal{V}^0(X, \mathbb{k}) \subseteq \mathcal{V}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{V}^n(X, \mathbb{k}).$$

This result puts some stringent constraints on the Betti numbers of such a space X , to wit, $b_p(X) > 0$ for $0 \leq p \leq n$, and $b_1(X) \geq n$.

1.3. The EPY property and propagation of resonance. A similar qualitative phenomenon also appeared in a paper of Eisenbud, Popescu and Yuzvinsky in [EPY03], who showed that the (depth 1) resonance varieties of complex hyperplane arrangement complements “propagate.” This result followed from another cohomological vanishing property, this time involving the cohomology ring itself.

If X is a space as above, we may regard its cohomology ring $A = H^\bullet(X, \mathbb{k})$ as a module over the exterior algebra E generated by $V := H^1(X, \mathbb{k})$. For hyperplane complements, these authors prove that A admits a linear, injective resolution as a graded E -module, after a shift of degrees. They express this condition in the elegant terms of the BGG correspondence, which shows it is equivalent to the statement that $H^p(A \otimes_{\mathbb{k}} S, d) = 0$ for all $p \neq n$, where S is the polynomial ring on V^* , and the differential d is induced by multiplication by the canonical element of $V \otimes V^*$. This complex and its vanishing property, it seems, should be regarded as an infinitesimal analogue of the equivariant cochain complex on the universal abelian cover of X together with its cohomological vanishing property. In view of these considerations, we will say that a space X has the *EPY property* if the shifted cohomology ring $A^*(n)$ has a linear free E -resolution, for some integer n .

Assume now that $\text{char}(\mathbb{k}) \neq 2$, or that $\text{char}(\mathbb{k}) = 2$ and $H_1(X, \mathbb{Z})$ has no 2-torsion. For each $a \in V$, then, left-multiplication by a defines a cochain complex, $(A, a \cdot)$. The (depth 1) *resonance varieties* of X over the field \mathbb{k} are the jump loci for the cohomology of these complexes; that is, $\mathcal{R}^p(X, \mathbb{k}) = \{a \in V \mid \dim_{\mathbb{k}} H^p(A, a \cdot) \neq 0\}$. The next result can be viewed as the infinitesimal analogue of Theorem 1.1.

Theorem 1.2. *Suppose X is a finite, connected CW-complex of dimension n with the EPY property over a field \mathbb{k} . Then the resonance varieties of X propagate: that is, for any $a \in A^1$, if $H^p(A, a \cdot) \neq 0$, then $H^q(A, a \cdot) \neq 0$ for all $p \leq q \leq n$, where $A = H^\bullet(X, \mathbb{k})$. Equivalently,*

$$\{0\} = \mathcal{R}^0(X, \mathbb{k}) \subseteq \mathcal{R}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{R}^n(X, \mathbb{k}).$$

Furthermore, if a space X as above admits a minimal CW-structure, then, for each $p \leq n$, the I -adic completion of the module $H^p(X, \mathbb{k}G^{\text{ab}})$ vanishes (Theorem 5.18).

1.4. Arrangements of submanifolds. Using our previous paper [DSY16], we show that complements of linear and elliptic hyperplane arrangements are abelian duality spaces. It follows from Theorem 1.1 that their characteristic varieties propagate. At least for linear arrangements, this phenomenon had been observed in examples and had been a matter of speculation. Linear hyperplane arrangements are also the first and motivating example of spaces with the EPY property.

Theorem 1.3. *Let \mathcal{A} be an arrangement in either $\mathbb{C}\mathbb{P}^n$ or $E^{\times n}$, and let $U = U(\mathcal{A})$ be its complement. In the first case, both the characteristic and the resonance varieties of U propagate. In the latter case, the characteristic varieties of U propagate.*

We make note of a special case. If \mathcal{A} is an arrangement of (linear) hyperplanes defined by the vanishing of a homogeneous polynomial f of degree m , the nonzero level sets of f are all diffeomorphic, and the *Milnor fiber* of \mathcal{A} , denoted $F(\mathcal{A})$, is the smooth affine hypersurface cut out by the equation $f = 1$. The Milnor fiber admits an action of the cyclic group $\mathbb{Z}/m\mathbb{Z}$: if h denotes a preferred generator, it is interesting to consider the action of h in homology. If ζ is an eigenvalue of h on $H_p(F(\mathcal{A}), \mathbb{C})$, let $H_p(F(\mathcal{A}), \mathbb{C})_\zeta$ denote the corresponding eigenspace. We show that Theorem 1.3 implies that

Corollary 1.4. *If \mathcal{A} is an arrangement of rank $n + 1$ and $H_p(F(\mathcal{A}), \mathbb{C})_\zeta \neq 0$ for some $p < n$, then $H_q(F(\mathcal{A}), \mathbb{C})_\zeta \neq 0$ as well, for all eigenvalues ζ and all $p \leq q \leq n$.*

Using the analysis from the previous section, we show that the Milnor fiber itself may be an abelian duality space. A sufficient condition is given by requiring that the action of h on $H_1(F(\mathcal{A}), \mathbb{Z})$ is trivial.

1.5. Polyhedral products. Another class of spaces where our approach works rather well is that of toric complexes, which are a special instance of generalized moment angle complexes, or polyhedral products, see e.g. [DS07, BBCG10]. Given a simplicial complex L with m vertices, the toric complex T_L is a subcomplex of the m -torus, whose k -dimensional subtori correspond to the k -vertex simplices of L . In particular, $\pi_1(T_L)$ is the right-angled Artin group associated to the 1-skeleton of L .

We characterize here which right-angled Artin groups are abelian duality groups, thus extending the work of Brady, Jensen and Meier [BM01, JM05]. It turns out that for right-angled Artin groups, the properties of being a duality group, being an abelian duality group, or having the EPY property are all equivalent to the Cohen–Macaulay property for the presentation’s flag complex.

Theorem 1.5. *Let L be an n -dimensional simplicial complex. Then the following are equivalent.*

- (1) *The toric complex T_L is an abelian duality space over \mathbb{k} (of dimension $n + 1$).*
- (2) *The cohomology algebra $H^*(T_L, \mathbb{k})$ has the EPY property.*
- (3) *The complex L is Cohen–Macaulay over \mathbb{k} .*

The Bestvina–Brady groups are subgroups of right-angled Artin groups that parallel the relation between the Milnor fiber and the complement of an arrangement. Using the behavior of the abelian duality property under fibrations again, we find that the Bestvina–Brady groups are also abelian duality groups if and only if they are duality groups, in parallel with the analysis of Davis and Okun [DO12].

1.6. Outline. For the readers’ convenience, we give a brief overview of the structure of the paper. Sections §2 and §3 are purely algebraic. We introduce cohomology jump loci and support loci in §2 for chain complexes and prove some preparatory lemmas. In §3, we recall the Bernstein–Gelfand–Gelfand correspondence in the context of jump loci, and we introduce resonance varieties.

The impatient reader is invited to proceed directly to §4, where we introduce the abelian duality property and compare it with the classical notion of a duality group, and then to §5, where we describe the various relationships between abelian duality, the EPY property, and propagation. We conclude by examining in depth two families of motivating examples: arrangements of submanifolds in §6, and polyhedral products in §7.

2. JUMP LOCI AND SUPPORT LOCI

2.1. Algebraic preliminaries I. We begin by singling out some general commutative algebra which we will use repeatedly. The reader may refer to [Eis95] for details. We will let \mathbb{k} denote a coefficient ring: unless specified otherwise, either $\mathbb{k} = \mathbb{Z}$ or a field. Let R be a \mathbb{k} -algebra (usually assumed to be commutative and Noetherian), and let $V = \mathfrak{m}\text{-Spec } R$. We will use the notational convention that, if C_\bullet is a chain complex, $C^i := C_{-i}$ for all $i \in \mathbb{Z}$.

Although the next two lemmas are well-known to experts, for lack of a reference, we include their proofs.

Lemma 2.1. *If R is a commutative, local \mathbb{k} -algebra, then for any finitely-generated R -module N , we have $\text{Tor}_i^R(N, \mathbb{k}) \neq 0$ for $0 \leq i \leq \text{pd}_R(N)$, and zero otherwise.*

Proof. Set $d = \text{pd}_R(N)$. Finitely-generated projective modules over a local ring are free, by Nakayama's lemma, so N has a free resolution of length d (where we allow $d = \infty$):

$$(1) \quad 0 \longleftarrow N \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_d \longleftarrow 0.$$

We may take this resolution to be minimal [Eis95, Corollary 19.5], in which case $\text{Tor}_i^R(N, \mathbb{k}) \cong F_i \otimes_R \mathbb{k}$. In particular, we see that $\text{Tor}_i^R(N, \mathbb{k})$ is nonzero precisely for $0 \leq i \leq d$. \square

Lemma 2.2. *Suppose M is an R -module and $\mathfrak{m} \in V$. Then*

$$\text{Tor}_i^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, \kappa(\mathfrak{m})) \neq 0 \Leftrightarrow \text{Tor}_i^R(M, \kappa(\mathfrak{m})) \neq 0,$$

for all $i \geq 0$.

Proof. Localization is exact, so $\text{Tor}_i^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, \kappa(\mathfrak{m})) = \text{Tor}_i^R(M, \kappa(\mathfrak{m}))_{\mathfrak{m}}$, for all $i \geq 0$. The implication “ \Rightarrow ” follows.

For the converse, we must check that if $x \in \text{Tor}_i^R(M, \kappa(\mathfrak{m}))$ is nonzero, then its image in the localization at \mathfrak{m} is also nonzero. It is sufficient to check that $rx = 0$ for $r \in R$ implies that either $r \in \mathfrak{m}$, or $x = 0$.

For this, let $F_{\bullet} \rightarrow M \rightarrow 0$ be a projective resolution of M . Suppose $c \in F_i \otimes_R \kappa(\mathfrak{m})$ is a representative for x . Then $rx = 0$ implies $rc = \partial(d)$ for some $d \in F_{i+1} \otimes_R \kappa(\mathfrak{m})$. If $r \notin \mathfrak{m}$, then $rs = 1 + m$ for some $s \in R$ and $m \in \mathfrak{m}$, since \mathfrak{m} is maximal. Then $\partial(sd) = (rs)c = c + mc = c$, and so $x = [c] = 0$, thereby completing the proof. \square

2.2. Algebraic preliminaries II. In this section, we briefly note some homological algebra special to Hopf algebras. We let R be a Hopf algebra with antipode $\iota: R \rightarrow R$, which we will always assume to be an involution. Our principal example will be the group algebra $R = \mathbb{k}[G]$, where the antipode is defined on the group elements by $\iota(g) = g^{-1}$.

If A is a right R -module, then we may also regard A as a left R -module by letting $r \cdot a := a \cdot \iota(r)$ for $r \in R$ and $a \in A$, and analogously if A is a left R -module. If we wish to emphasize the point, we will write $\iota(A)$ in place of A . Even when R is commutative, then, we preserve the distinction between left and right modules. If A and B are left R -modules, then $A \otimes_{\mathbb{k}} B$ and $\text{Hom}_{\mathbb{k}}(A, B)$ are left R -modules via the coproduct and, respectively, the action $(rf)(a) = f(\iota(r) \cdot a)$, for $f \in \text{Hom}_{\mathbb{k}}(A, B)$, all $r \in R$, and $a \in A$.

Lemma 2.3. *Let R be a Hopf algebra over \mathbb{k} . Suppose D is a right R -module and A is a left R -module.*

- (1) *If D is flat over \mathbb{k} , then $\text{Tor}_{\bullet}^R(D, A) = \text{Tor}_{\bullet}^R(\mathbb{k}, D \otimes_{\mathbb{k}} A)$.*
- (2) *If D is projective over \mathbb{k} , then $\text{Ext}_{\bullet}^R(D, A) = \text{Ext}_{\bullet}^R(\mathbb{k}, \text{Hom}_{\mathbb{k}}(D, A))$.*

Proof. For the first statement, we claim it is enough to check that

$$(2) \quad D \otimes_R A \cong \mathbb{k} \otimes_R (D \otimes_{\mathbb{k}} A)$$

for any A . Indeed, suppose Q_{\bullet} is a projective resolution of A . It is easily checked that if B is a flat R -module, then so is $D \otimes_{\mathbb{k}} B$; then $D \otimes_{\mathbb{k}} Q_{\bullet}$ is a flat resolution of A , and the result follows.

To verify (2), let η and ε denote the unit and counit, respectively. We check that the \mathbb{k} -bilinear map induced by $(c, d \otimes a) \mapsto \eta(c)d \otimes a$ is R -bilinear. For this, we note $(1 \cdot r, d \otimes a) \mapsto$

$\eta\varepsilon(r)d \otimes a$. On the right, we compute with Sweedler notation, as in [Sw69]:

$$\begin{aligned} (1, r \cdot (d \otimes a)) &\mapsto \sum dt(r_{(1)}) \otimes r_{(2)}a \\ &= \sum dt(r_{(1)})r_{(2)} \otimes a \\ &= \eta\varepsilon(r)d \otimes a. \end{aligned}$$

For the second statement, we check that

$$(3) \quad \mathrm{Hom}_R(D, A) \cong \mathrm{Hom}_R(\mathbb{k}, \mathrm{Hom}_{\mathbb{k}}(D, A)),$$

using a similar argument, which we omit. \square

2.3. Jump loci of chain complexes. Various notions of jump loci for rank-1 local systems, resonance varieties of graded commutative algebras, and support varieties of Alexander invariants of spaces are all instances of the following basic notions. Here, we assume that \mathbb{k} is an algebraically closed field, and R is commutative.

Definition 2.4. If C_\bullet is a chain complex of finitely-generated R -modules, we define its *homology jump loci* as follows:

$$(4) \quad \mathcal{V}_{i,d}(C_\bullet) := \{ \mathfrak{m} \in V : \dim_{\kappa(\mathfrak{m})} H_i(C \otimes_R \kappa(\mathfrak{m})) \geq d \},$$

for all integers $d \geq 0$ and all i . Here, $\kappa(\mathfrak{m}) := R/\mathfrak{m}$.

Definition 2.5. The *homology support loci* of C_\bullet are, by definition,

$$(5) \quad \mathcal{W}_{i,d}(C_\bullet) := \mathrm{supp} \left(\bigwedge^d H_i(C_\bullet) \right) \subseteq V.$$

Clearly, the sets $\mathcal{W}_{i,d}(C_\bullet)$ depend only on the quasi-isomorphism class of the chain complex C_\bullet .

In both cases, if we wish to regard C_\bullet as a cochain complex, we will write $\mathcal{W}^{i,d}(C^\bullet) := \mathrm{supp}(\bigwedge^d H^i(C^\bullet))$, and use the notation $\mathcal{V}^{i,d}(C^\bullet)$ analogously. We abbreviate $\mathcal{V}_i(C_\bullet) := \mathcal{V}_{i,1}(C_\bullet)$, and likewise for $\mathcal{W}_i(C_\bullet)$, etc.

Remark 2.6. By construction, the sets $\mathcal{W}_{i,d}(C_\bullet)$ are Zariski-closed subsets of $V = \mathrm{m}\text{-Spec } R$. On the other hand, there are chain complexes C_\bullet for which the sets $\mathcal{V}_i(C_\bullet)$ are not Zariski-closed, see e.g. [PS14, Example 2.4]. Nevertheless, if C_\bullet is a chain complex of *free* (finitely-generated) R -modules, then all the jump loci $\mathcal{V}_{i,d}(C_\bullet)$ are Zariski-closed. \diamond

In view of the above remark, we need to modify the above definition of jump loci in the case when one of the R -modules C_i is not free. By a result of Mumford (see [Ha77, III.12.3]), there is a chain complex F_\bullet of free R -modules which is quasi-isomorphic to C_\bullet ; moreover, if C_\bullet is bounded below, then F_\bullet may be chosen to be bounded below, too. Finally, as shown by Budur and Wang [BW15], the varieties $\mathcal{V}_{i,d}(C_\bullet) := \mathcal{V}_{i,d}(F_\bullet)$ depend only on C_\bullet , and not on the choice of F_\bullet .

The various notions defined above are different, in general. Nevertheless, as noted in [PS14], there is a spectral sequence which allows us to relate the jump loci $\mathcal{V}_i(C_\bullet)$ and the support loci $\mathcal{W}_i(C_\bullet)$. For completeness, we reprove this result, in a slightly more general context.

Proposition 2.7 ([PS14], Cor. 4.3). *Suppose that C_\bullet is bounded below. Then, for each $j \in \mathbb{Z}$,*

$$(6) \quad \bigcup_{i \leq j} \mathcal{V}_i(C_\bullet) = \bigcup_{i \leq j} \mathcal{W}_i(C_\bullet).$$

Proof. By the above discussion, we may assume $C.$ is a chain complex of free modules. Since $C.$ is bounded below, there is a spectral sequence starting at $E_{pq}^2 = \mathrm{Tor}_p^R(H_q C, \kappa(\mathfrak{m}))$ and converging to $H_{p+q}(C. \otimes_R \kappa(\mathfrak{m}))$ for all $\mathfrak{m} \in V$. So if $\mathfrak{m} \in \mathcal{V}_j(C.)$ for some j , it must be the case that $E_{p,j-p}^2 \neq 0$ for some $p \geq 0$. It follows from Lemma 2.2 that $(H_{j-p} C)_{\mathfrak{m}} \neq 0$: i.e., $\mathfrak{m} \in \mathcal{W}_{j-p}(C.)$, which gives the forward containment in (6).

On the other hand, suppose j is the least index for which $\mathfrak{m} \in \mathcal{W}_j(C.)$. By Lemma 2.2 again, we have that $(E_{pq}^2)_{\mathfrak{m}} = 0$ for all $q < j$ (and $p < 0$). By hypothesis, $E_{0j}^2 \neq 0$, so $E_{0j}^\infty \neq 0$ as well. That is, $\mathcal{W}_j(C.) \setminus \bigcup_{i < j} \mathcal{W}_i(C.)$ is included in $\mathcal{V}_j(C.)$, from which the other containment follows. \square

2.4. Jump loci and cohomological vanishing. We now consider a special case, the jump loci of projective resolutions. If $C.$ is a chain complex of R -modules, for $n \in \mathbb{Z}$ let $C[n]_i := C_{n+i}$ the shifted complex. In this case, the jump loci are nested.

Proposition 2.8. *Suppose that R is commutative and $C.$ is a complex of finitely-generated projective modules for which $C_i = 0$ for $i < n$, for some n , and $H_i(C) = 0$ except for $i = n$. Then*

$$(7) \quad V \supseteq \mathcal{V}_n(C.) \supseteq \mathcal{V}_{n+1}(C.) \supseteq \mathcal{V}_{n+2}(C.) \supseteq \cdots .$$

We will refer to the phenomenon (7), when it occurs, as *propagation* of homology jump loci.

Proof. Clearly it is sufficient to consider the case $n = 0$. Let $D = H_0(C)$. Suppose that $\mathfrak{m} \in \mathcal{V}_i(C.)$ for some $i > 0$ and j is an integer for which $n \leq j \leq i$. Since $C.$ is a resolution,

$$\begin{aligned} \mathfrak{m} \in \mathcal{V}_i(C.) &\Leftrightarrow \mathrm{Tor}_i^R(D, \kappa(\mathfrak{m})) \neq 0 \\ &\Leftrightarrow \mathrm{Tor}_i^{R_{\mathfrak{m}}}(D_{\mathfrak{m}}, \kappa(\mathfrak{m})) \neq 0 \\ &\Rightarrow \mathrm{Tor}_j^R(D, \kappa(\mathfrak{m})) \neq 0 \\ &\Rightarrow \mathfrak{m} \in \mathcal{V}_j(C.), \end{aligned}$$

by Lemma 2.2 and then Lemma 2.1. The claim follows. \square

In our applications, the module D in the proof above will be the dualizing module.

2.5. Propagation and duality. If C is a projective resolution, we can also describe jump loci in terms of homology support loci. Again, suppose R is a \mathbb{k} -Hopf algebra with antipode ι . Let $-^* := \mathrm{Hom}_R(-, \mathbb{k})$ and $-^\vee := \mathrm{Hom}_R(-, R)$.

Proposition 2.9. *Suppose $C.$ is a chain complex of finitely-generated, projective right R -modules over a field \mathbb{k} . Then*

$$(8) \quad \iota(\mathcal{V}_{i,d}(C.)) = \mathcal{V}^{i,d}(C^\vee),$$

for all integers i, d .

Proof. Since $C.$ is finitely-generated and projective, for any right R -module A , the natural map

$$(9) \quad \mathrm{Hom}_R(C., R) \otimes_R \iota(A) \longrightarrow \mathrm{Hom}_R(C., A)$$

is an isomorphism. Using the Hom-tensor adjunction, for any $\mathfrak{m} \in V = \mathfrak{m}\text{-Spec } R$,

$$(10) \quad \begin{aligned} (C_\bullet \otimes_R \kappa(\mathfrak{m}))^* &\cong \text{Hom}_R(C_\bullet, \kappa(\mathfrak{m})^*) \\ &\cong C^\vee \otimes_R \kappa(\iota(\mathfrak{m})) \end{aligned}$$

via the isomorphism above. Since \mathbb{k} is a field, the Universal Coefficients Theorem implies this is an isomorphism in homology, which completes the proof. \square

Proposition 2.10. *Suppose that $C_\bullet[n]$ is a finitely-generated projective resolution for some $n \in \mathbb{Z}$. Then*

$$\mathcal{V}_k(C_\bullet) = \mathcal{V}_{k+1}(C_\bullet) \cup \iota(\mathcal{W}^k(C^\vee))$$

for all $k \geq n$.

Proof. We apply Proposition 2.7 to C^\vee , since (by hypothesis) C^\vee is also bounded below. For all integers k ,

$$\begin{aligned} \bigcup_{i \geq k} \iota(\mathcal{W}^i(C^\vee)) &= \bigcup_{i \geq k} \iota(\mathcal{V}^i(C^\vee)) \\ &= \bigcup_{i \geq k} \mathcal{V}_i(C_\bullet), && \text{by Proposition 2.9,} \\ &= \mathcal{V}_k(C_\bullet) && \text{provided } k \geq n, \end{aligned}$$

using Proposition 2.8. The result follows. \square

3. THE BGG CORRESPONDENCE, KOSZUL MODULES, AND RESONANCE

3.1. The BGG correspondence. Let V be a finite-dimensional vector space over a field \mathbb{k} . Let V^* denote the dual vector space. The Bernstein–Gelfand–Gelfand correspondence is an explicit equivalence of bounded derived categories of graded modules over the exterior algebra $E := \bigwedge V$ and over the symmetric algebra $S := \text{Sym } V^*$.

Following the exposition in [Eis05], we assign the nonzero elements of V^* degree 1; then $S_1 = V^*$ and $E_{-1} = E^1 = V$. If M is a graded module and $r \in \mathbb{Z}$, we denote a degree shift by letting $M(r)_i := M_{r+i}$; then $M(r)^i = M^{i-r}$.

Let \mathbf{L} denote the functor from the category of graded E -modules to the category of linear free complexes over S , defined as follows: for a graded E -module P , $\mathbf{L}(P)$ is the complex

$$(11) \quad \cdots \longrightarrow P_i \otimes_{\mathbb{k}} S \xrightarrow{d_i} P_{i-1} \otimes_{\mathbb{k}} S \longrightarrow \cdots,$$

with an S -linear differential d induced by left-multiplication by the canonical element $\omega \in V \otimes V^*$. We note a useful fact about the functor \mathbf{L} .

Lemma 3.1 ([Eis05], Ex 7F7). *If P is a finitely-generated E -module, then $\mathbf{L}(P)^\vee = \mathbf{L}(P^*)$.*

3.2. Resonance varieties. Here is another important example of (co)homology jump loci. Let P be a finitely-generated E -module. For each $a \in E^1 = V$, we note that $a^2 = 0$, so left-multiplication by a defines a cochain complex

$$(12) \quad (P, a\cdot): \cdots \longrightarrow P^{i-1} \xrightarrow{a} P^i \xrightarrow{a} P^{i+1} \longrightarrow \cdots.$$

We note that the complex $(P, a\cdot)$ is a specialization of the construction (11): that is, if we make the identification $V = \mathfrak{m}\text{-Spec}(S)$, it follows easily that $(P, a\cdot) = \mathbf{L}(P) \otimes_S \kappa(a)$. This

makes $\mathbf{L}(P)$ a “universal complex” that parameterizes the complexes $(P, a\cdot)$. The corresponding jump loci in this case are known as ‘resonance’ varieties.

Definition 3.2. If P is a finitely-generated graded E -module, its *resonance varieties* are defined to be

$$(13) \quad \begin{aligned} \mathcal{R}^{i,d}(P) &:= \mathcal{V}^{i,d}(\mathbf{L}(P)) \\ &= \{a \in V : \dim_{\mathbb{k}} H^i(P, a\cdot) \geq d\}. \end{aligned}$$

As usual, let $\mathcal{R}^i(P) := \mathcal{R}^{i,1}(P)$.

Clearly, these sets are homogeneous algebraic subvarieties of the affine space V . Moreover, if $P^i = 0$, then $\mathcal{R}^{i,d}(P) = \emptyset$, for all $d > 0$. Our convention is to index these jump loci cohomologically, since this is the natural choice for examples in the literature.

3.3. Propagation of resonance. In this section, we recall some equivalent conditions under which the BGG correspondence becomes particularly simple.

Theorem 3.3 ([Eis05], Thm. 7.7). *For a finitely-generated graded E -module P with $P_i = 0$ for $i < 0$, the following are equivalent.*

- (1) P^* has a linear free resolution.
- (2) $H_i(\mathbf{L}(P)) = 0$ for $i \neq 0$.

Definition 3.4. If P satisfies the hypotheses above, the E -module P^* is called a *Koszul module*. In this case, we let $F(P) := H_0(\mathbf{L}(P))$, an S -module.

With this, we obtain a special case of Proposition 2.8.

Proposition 3.5. *Suppose that P is an E -module satisfying one of the equivalent conditions of Theorem 3.3. Then*

$$(14) \quad V \supseteq \mathcal{R}^0(P) \supseteq \mathcal{R}^{-1}(P) \supseteq \mathcal{R}^{-2}(P) \supseteq \dots$$

Proof. We simply invoke Proposition 2.8, with $C = \mathbf{L}(P)$, and $n = 0$. □

Specializing further, consider the case of a cyclic, graded E -module, $A := E/I$. Following the development in [EPY03], we make the following definition.

Definition 3.6. A cyclic, graded E -module A has the *EPY property* if $A^*(n)$ is a Koszul module for some integer n . If X is a finite-type CW-complex of dimension n and $A = H^*(X, \mathbb{k})$, we say X has the EPY property.

If X has the EPY property, we note that the integer n must be the socle degree of A , regarded as a \mathbb{k} -algebra. Another immediate consequence is the following.

Proposition 3.7. *If X has the EPY property over \mathbb{k} , then $\dim_{\mathbb{k}} H^i(X, \mathbb{k}) \geq \binom{n}{i}$.*

Proof. By Theorem 3.3, $\mathbf{L}(A(-n))$ is a linear, free resolution of the S -module $F(A(-n))$, so $\dim_{\mathbb{k}} A^{n-i}$ is its i th Betti number. This lower bound on Betti numbers in a linear resolution is due to Herzog and Kühl [HK84]. □

Theorem 3.8. *Suppose A has the EPY property, with socle degree n . Then the resonance varieties of A propagate:*

$$\mathcal{R}^i(A) \subseteq \mathcal{R}^{i+1}(A)$$

for $0 \leq i < n$.

Proof. We apply Proposition 3.5 to the module $P = A(-n)$, and note that $\mathcal{R}^i(A(-n)) = \mathcal{R}^{i+n}(A)$, for $i \in \mathbb{Z}$. \square

Example 3.9. The free E -module of rank 1 has the EPY property, since $E^*(n) \cong E$; thus the n -torus $T = (S^1)^{\times n}$ is a space with the EPY property. It is easily seen that $\mathcal{R}^i(E) = \{0\}$ for $0 \leq i \leq n$. \diamond

Note that S is a Hopf algebra with antipode defined in degree 1 by $x \mapsto -x$. Since the resonance varieties are homogeneous, Proposition 2.9 simplifies to the statement that

$$(15) \quad \begin{aligned} \mathcal{R}^{i,d}(P) &= \mathcal{V}_{i,d}(\mathbf{L}(P)^\vee) \\ &= \mathcal{R}_{i,d}(P^*). \end{aligned}$$

Let us also consider the relationship between the jump loci and support loci in the context of propagation. If P^* is a Koszul module, then by Theorem 3.3, $\mathbf{L}(P)$ resolves $F(P)$. By Lemma 3.1, then,

$$(16) \quad \begin{aligned} H^i(\mathbf{L}(P^*)) &= H^i(\mathbf{L}(P)^\vee) \\ &= \text{Ext}_S^i(F(P), S). \end{aligned}$$

Consequently, Proposition 2.10 takes the following form.

Proposition 3.10. *Suppose that P^* is a Koszul E -module. Then*

$$\mathcal{R}^{-k}(P) = \mathcal{R}^{-k-1}(P) \cup \text{supp Ext}_S^k(F(P), S)$$

for all $k \geq 0$.

In the special case where A is a cyclic E -module with socle degree n , this says $\mathcal{R}^k(A) = \mathcal{R}^{k-1}(A) \cup \text{supp Ext}_S^{n-k}(F(A), S)$, for $0 \leq k \leq n$. We refer to Example 6.3 to see this illustrated explicitly.

4. DUALITY AND ABELIAN DUALITY SPACES

4.1. Duality groups. We now recall a well-known notion, due to Bieri and Eckmann [BE73]. Let G be a group, and let $\mathbb{Z}G$ be its group-ring. For simplicity, we will assume G is of type FP, i.e., there is a finite resolution $P_\bullet \rightarrow \mathbb{Z} \rightarrow 0$ of the trivial $\mathbb{Z}G$ -module \mathbb{Z} by finitely generated projective $\mathbb{Z}G$ -modules.

Definition 4.1. An FP-group G is called a *duality group* of dimension n if $H^p(G, \mathbb{Z}G) = 0$ for $p \neq n$ and $H^n(G, \mathbb{Z}G)$ is non-zero and torsion-free.

If G satisfies the above condition, the abelian group $C = H^n(G, \mathbb{Z}G)$, viewed as a (right) module over $\mathbb{Z}G$, is called the *dualizing module* of G . Then n is the cohomological dimension of G , and the hypotheses imply that the complex

$$(17) \quad 0 \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \dots \longrightarrow P_n^\vee \longrightarrow C \longrightarrow 0$$

is a finitely generated projective resolution of C . Using the identification (9), for all i we have

$$(18) \quad \begin{aligned} H^i(G, A) &\cong \text{Tor}_{n-i}^{\mathbb{Z}G}(C, A) \\ &\cong H_{n-i}(G, C \otimes_{\mathbb{Z}} A), \end{aligned}$$

by Lemma 2.3.

By a celebrated theorem of Stallings and Swan, the duality groups of dimension 1 are precisely the finitely generated free groups. It is also known that torsion-free, one-relator

groups are duality groups of dimension 2. The class of duality groups is closed under extensions and passing to finite-index subgroups. On the other hand, a free product of duality groups is not a duality group, unless all the factors are free, cf. [Br82].

Remark 4.2. If we replace \mathbb{Z} by a principal ideal domain \mathbb{k} in Definition 4.1, we will say G is a *duality group over \mathbb{k}* . We use the same terminology in the definitions that follow. We will consider only the cases where $\mathbb{k} = \mathbb{Z}$ or a field. Clearly, if G is a duality group, then it is a duality group over any choice of \mathbb{k} . \diamond

In [BE73, Theorem 6.2] Bieri and Eckmann provide a geometric criterion for a group G to be a duality group. Namely, suppose G admits as classifying space a compact, connected, orientable manifold-with-boundary M of dimension m , and suppose $\partial\widetilde{M}$, the boundary of the universal cover of M , has the integral homology of a wedge of q -spheres. Then G is a duality group of dimension $n = m - q - 1$, with dualizing module $C = H_q(\partial\widetilde{M}, \mathbb{Z})$.

Of note is the situation when the dualizing module, $C = H^n(G, \mathbb{Z}G)$, is the trivial $\mathbb{Z}G$ -module \mathbb{Z} . In this case, G is an (orientable) *Poincaré duality group* of dimension n —for short, a PD_n -group. For instance, if G admits a classifying space which is a closed, orientable, manifold M of dimension n (this is the case $\partial M = \emptyset$ and $q = -1$ from above), then G belongs to this class. If G is a duality group that is not a Poincaré duality group, then the dualizing module C contains no non-zero $\mathbb{Z}G$ -submodule which is finitely generated over \mathbb{Z} , cf. [Fa75].

4.2. Bieri–Eckmann duality in context. Bieri–Eckmann duality for representations of discrete groups bears a formal resemblance to Serre duality for coherent sheaves on projective, Cohen–Macaulay schemes. Since the latter can be understood in terms of Verdier duality, as explained in [Har66], it is natural to examine the former in this light as well. A general framework for such duality theories is provided by Boyarchenko and Drinfeld [BD13].

To start, for any FP-group G , there is a duality isomorphism in the derived category that takes on the stronger form described in the previous section in the special case when G is a duality group. That is, for any coefficient ring \mathbb{k} , let $R = \mathbb{k}G$, and let $\omega_G := \mathbf{R}\text{Hom}(\mathbb{k}, R)$. Since \mathbb{k} has a finite projective resolution, $\omega_G \in \mathbf{D}^b(\text{mod-}R)$. Since that resolution can be taken to be finitely generated, and for finitely generated projective modules P there is an isomorphism $\text{Hom}_R(P, R) \otimes_R - \cong \text{Hom}_R(P, -)$, we have an isomorphism of functors on $\mathbf{D}^b(\text{mod-}R)$ generalizing the duality group isomorphism (18):

$$(19) \quad \begin{aligned} \mathbf{R}\text{Hom}_R(\mathbb{k}, -) &\cong \omega_G \otimes_R^{\mathbf{L}} - \\ &\cong \mathbb{k} \otimes_R^{\mathbf{L}} (\omega_G \otimes_{\mathbb{k}}^{\mathbf{L}} -), \end{aligned}$$

using Lemma 2.3 again. In this language, G is a \mathbb{k} -duality group of dimension n if and only if ω_G is quasiisomorphic to a complex concentrated in cohomological degree n .

To move beyond this purely algebraic setting, let us now assume that G is a FL group; that is, G admits a compact classifying space BG , with universal cover EG . Let \mathbb{k} be a field, and take [Ive86, KS90, Dim04] as references for sheaf theory. As observed in [Br82, p. 209], then $H^*(G, \mathbb{k}[G]) = H_c^*(EG, \mathbb{k})$. Let f denote the constant function on EG . Then Verdier duality implies an isomorphism in cohomology

$$(20) \quad H_c^p(EG, \mathbb{k})^\vee \cong \mathbb{H}^{-p}(EG, f^! \mathbb{k})$$

for all p , where $f^! \mathbb{k} \in \mathbf{D}^b(EG)$ is the dualizing complex on EG . That is, G is a \mathbb{k} -duality group of dimension n if and only if the hypercohomology (20) vanishes for $p \neq n$. In the case where $f^! \mathbb{k}$ is a complex of local systems (for example, if EG is a manifold), the hypercohomology

spectral sequence degenerates, so G is a \mathbb{k} -duality group if and only if $f^!\mathbb{k}$ is represented by a complex of sheaves which is nonzero only in cohomological dimension n .

Remark 4.3. In general, the dualizing complex is constructible but not a local system, in which case the property that $f^!\mathbb{k} \cong \omega[n]$ for some sheaf ω on EG does not imply that G is a duality group. Indeed, suppose G admits as classifying space an aspherical, orientable manifold-with-boundary M^m , and let $j: \widetilde{M} \setminus \partial\widetilde{M} \rightarrow \widetilde{M}$ be the inclusion map. Then $f^!\mathbb{k} = j_!\mathbb{k}[m]$, where \mathbb{k} is the constant sheaf: see, e.g., [Ive86, p. 298]. Yet G need not be a duality group: for instance, take $G = F_2 * \mathbb{Z}$, and let $BG = M^4$ be a thickening of $T^2 \vee S^1$ in \mathbb{R}^4 . \diamond

Finally, as noted by Bieri and Eckmann, there is an intriguing analogy in the context of profinite groups between their notion of duality groups and what Verdier calls ‘‘Cohen–Macaulay groups’’ in the last chapter of [Se62]. This analogy was explored more fully by Pletsch in [Pl80]. It would be interesting to know if there could be an arithmetic analogue of abelian duality and its consequences for local systems considered in Section 5.

4.3. Duality spaces. The notion introduced in Definition 4.1 can be extended from groups to spaces. One possible definition of a duality space is given in [BE73, § 6.2]. For our purposes, we adopt a slightly different definition, inspired by recent work of Davis, Januszkiewicz, Leary, and Okun [DLJO11].

Let X is a path-connected space with the homotopy type of a CW-complex. We shall assume X is of finite-type, i.e., it has finitely many cells in each dimension. Without loss of generality, we may assume X has a single 0-cell, say, $x_0 \in X$. Let $G = \pi_1(X, x_0)$ be the fundamental group of X .

Definition 4.4. A space X as above is called a *duality space* of dimension n if $H^p(X, \mathbb{Z}G) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G)$ is non-zero and torsion-free.

As we noted in the aspherical case, this is equivalent to the statement that the compactly supported cohomology of the universal cover, \widetilde{X} , is concentrated in a single dimension (where it is torsion-free). Clearly, if X is aspherical, then its fundamental group is a duality group. Generalizing the discussion above slightly shows that, if X is a duality space of dimension n and $C = H^n(X, \mathbb{Z}G)$, then

$$(21) \quad H^i(X, A) \cong H_{n-i}(G, C \otimes_{\mathbb{Z}} A)$$

for all $i \geq 0$ and $\mathbb{Z}G$ -modules A .

4.4. Abelian duality spaces. Let X^{ab} be the universal abelian cover of X . The fundamental group of X^{ab} is G' , the commutator subgroup of G , while the group of deck transformations is the abelianization, $G^{\text{ab}} = G/G'$. We introduce now a variation on the above definition, which essentially replaces the universal cover by the universal abelian cover.

Definition 4.5. The space X is called an *abelian duality space* of dimension n if X is homotopy equivalent to a finite, connected CW-complex of dimension n , $H^p(X, \mathbb{Z}G^{\text{ab}}) = 0$ for $p \neq n$, and $H^n(X, \mathbb{Z}G^{\text{ab}})$ is non-zero and torsion-free. If in addition X is aspherical, then we will call $G := \pi_1(X, *)$ an *abelian duality group* of dimension n .

In that case, the group $D = H^n(X, \mathbb{Z}G^{\text{ab}})$, viewed as a module over $\mathbb{Z}G^{\text{ab}}$, is called the *dualizing module* of X . We shall sometimes refer to these spaces as ab-duality spaces. Just as in §4.2, we can reformulate this notion as a property of the universal abelian cover. Once again, let f denote the constant map on $\widetilde{X}^{\text{ab}}$: then X is an abelian duality space if and only if $\mathbb{H}^p(\widetilde{X}^{\text{ab}}, f^!\mathbb{Z}) = 0$ for $p \neq -n$, and is nonzero and torsion-free for $p = -n$.

As an example, finitely generated free groups are abelian duality groups. On the other hand, surface groups of genus at least 2 are *not* abelian duality groups (see Example 5.8), though of course they are (Poincaré) duality groups. Next, we give an example (suggested by Ian Leary) of an abelian duality group which is not a duality group:

Example 4.6. Let $H = \langle x_1, \dots, x_4 \mid x_1^{-2}x_2x_1x_2^{-1}, \dots, x_4^{-2}x_1x_4x_1^{-1} \rangle$ be Higman's acyclic group, and let $G = \mathbb{Z}^2 * H$. Then G is an abelian duality group (of dimension 2), but not a duality group. \diamond

Of note is the situation when the dualizing module, $D = H^n(G, \mathbb{Z}G^{\text{ab}})$, is the trivial $\mathbb{Z}G^{\text{ab}}$ -module \mathbb{Z} . Let us call such a group a *Poincaré abelian duality group* of dimension n . Such groups (with $n = 2$) were studied by Fenn and Sjerve in [FSS2]. We extend the notion as before to spaces.

Question 4.7. Suppose G is an abelian duality group that is not a Poincaré abelian duality group. Does the dualizing module D contain any non-zero $\mathbb{Z}G^{\text{ab}}$ -submodule which is finitely generated over \mathbb{Z} ?

As the name is intended to suggest, abelian duality spaces have a (co)homological property which is analogous to the classical one of (18). For convenience, let $R = \mathbb{Z}G^{\text{ab}}$.

Proposition 4.8. *Suppose X is an abelian duality space of dimension n with dualizing module D , and A is a (left) R -module. Then*

$$(22) \quad H^i(X, A) \cong \text{Tor}_{n-i}^R(D, A) \cong H_{n-i}(G^{\text{ab}}, D \otimes_{\mathbb{Z}} A)$$

for all $i \geq 0$. If, moreover, D is a free abelian group, then

$$(23) \quad H_i(X, A) \cong H^{n-i}(G^{\text{ab}}, \text{Hom}_{\mathbb{Z}}(D, A)).$$

Proof. By hypothesis, $C^*(\tilde{X}^{\text{ab}})[n]$ is a finitely-generated complex of free R -modules, so we may identify $\text{Hom}_R(C^*(\tilde{X}^{\text{ab}}), A)$ with $C^*(\tilde{X}^{\text{ab}}) \otimes_R A$. Since $C^*(\tilde{X}^{\text{ab}})[n]$ is a resolution of D , we may now apply Lemma 2.3. The second claim is similar. \square

One consequence is that the Poincaré abelian duality property is rather rare.

Corollary 4.9. *Suppose X is both a Poincaré duality space and an abelian duality space. Then X is a Poincaré abelian duality space. Moreover, for such X the commutator subgroup of $\pi_1(X)$ is perfect.*

Proof. Using the first property, and the isomorphism (21), we see that $D = H^n(X, \mathbb{Z}G^{\text{ab}}) = H_0(X, \mathbb{Z}G^{\text{ab}}) = \mathbb{Z}$, the trivial module, and this proves the first claim.

Now we assume X is a Poincaré abelian duality space. Using now the isomorphism (23), we see that

$$\begin{aligned} H_p(X, \mathbb{Z}G^{\text{ab}}) &= H^{n-p}(G^{\text{ab}}, \mathbb{Z}G^{\text{ab}}) \\ &= 0 \text{ for all } p \neq 0. \end{aligned}$$

In particular, $G'/G'' \cong H_1(X, \mathbb{Z}G^{\text{ab}}) = 0$, and the second claim is proved. \square

Example 4.10. Let $H_{\mathbb{R}}$ be the algebraic group of 3×3 unipotent matrices with entries in \mathbb{R} , and let $H_{\mathbb{Z}}$ be subgroup of integral matrices. The quotient space, $M = H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a closed, orientable, aspherical 3-manifold, known as the Heisenberg nilmanifold. Its fundamental group G is the free, 2-step nilpotent group of rank 2. Since G is a Poincaré duality group of dimension 3, it cannot be an abelian duality group. We will return to this group in Example 4.16. \diamond

4.5. Abelian duality and extensions. In [BE73, Thm. 3.5], Bieri and Eckmann showed that the class of duality groups is closed under extensions. In this section, we note that some analogous results hold for abelian duality groups and spaces.

Suppose $F \rightarrow E \rightarrow B$ is a fibration sequence of path-connected CW-complexes. The exact sequence of low-degree terms in the Serre spectral sequence reads as follows:

$$(24) \quad H_2(E, \mathbb{Z}) \longrightarrow H_2(B, \mathbb{Z}) \xrightarrow{d} H_1(F, \mathbb{Z})_Q \longrightarrow H_1(E, \mathbb{Z}) \longrightarrow H_1(B, \mathbb{Z}) \longrightarrow 0,$$

where $Q = \pi_1(B)$.

Definition 4.11. We say that the sequence $F \rightarrow E \rightarrow B$ is *ab-exact* if

- (1) Q acts trivially on $H_1(F, \mathbb{Z})$; and
- (2) the map $d: H_2(B, \mathbb{Z}) \rightarrow H_1(F, \mathbb{Z})$ is zero.

In the presence of condition (1), condition (2) is equivalent to the exactness of the sequence $0 \rightarrow N^{\text{ab}} \rightarrow G^{\text{ab}} \rightarrow Q^{\text{ab}} \rightarrow 0$, where $N = \pi_1(F)$ and $G = \pi_1(E)$.

Remark 4.12. For fibrations over S^1 , the second condition follows from the first, since the map d in (24) is zero for dimensional reasons. In this case, ab-exactness is equivalent to trivial monodromy action of the base on the first homology group of the fiber. \diamond

Although the definition is clearly quite restrictive, we will see that some interesting group extensions in §6 and §7 are indeed ab-exact.

Proposition 4.13. *Suppose $F \rightarrow E \rightarrow B$ is an ab-exact fibration of path-connected, finite-type CW-complexes.*

- (1) *If F and B are ab-duality spaces of dimensions n and r , respectively, then E is an ab-duality space of dimension $n + r$.*
- (2) *If F and E are ab-duality spaces of dimensions n and $n + r$, respectively, and $\dim B = r$, then B is an ab-duality space of dimension r .*
- (3) *If E and B are ab-duality spaces of dimensions $n + r$ and r , respectively, and $\dim F = n$, then F is an ab-duality space of dimension n .*

Proof. Set $Q = \pi_1(B)$, $G = \pi_1(E)$, and $N = \pi_1(F)$. The Serre spectral sequence has

$$E_2^{pq} = H^p(B, H^q(F, \mathbb{Z}[G^{\text{ab}}])) \Rightarrow H^{p+q}(E, \mathbb{Z}[G^{\text{ab}}]).$$

Clearly, the action of $N = \pi_1(F)$ on $\mathbb{Z}[G^{\text{ab}}]$ factors through N^{ab} . Furthermore, exactness of (24) insures that $\mathbb{Z}[G^{\text{ab}}] \cong \mathbb{Z}[Q^{\text{ab}}] \otimes_{\mathbb{Z}} \mathbb{Z}[N^{\text{ab}}]$, as a $\mathbb{Z}[N^{\text{ab}}]$ -module. By hypothesis, F is of finite-type, so $H^*(F, -)$ commutes with colimits, giving

$$(25) \quad E_2^{pq} \cong H^p(B, \mathbb{Z}[Q^{\text{ab}}] \otimes H^q(F, \mathbb{Z}[N^{\text{ab}}])).$$

To prove (1), assume F and B are abelian duality spaces with dualizing modules D_F and D_B , respectively. Then $E_2^{pq} = 0$ unless $q = n$, in which case

$$E_2^{pn} \cong H^p(B, \mathbb{Z}[Q^{\text{ab}}] \otimes_{\mathbb{Z}} D_F).$$

By Definition 4.11(1), the action of Q on $\mathbb{Z}[N^{\text{ab}}]$ is trivial, so D_F is also trivial as a Q -module. Again, B is of finite-type, so

$$E_2^{pn} \cong H^p(B, \mathbb{Z}[Q^{\text{ab}}]) \otimes_{\mathbb{Z}} D_F,$$

which is zero unless $p = r$, where $E_2^{rn} \cong D_B \otimes_{\mathbb{Z}} D_F$, and the spectral sequence gives $H^p(E, \mathbb{Z}[G^{\text{ab}}]) \cong D_B \otimes_{\mathbb{Z}} D_F$ for $p = r + n$, and zero otherwise. This is torsion-free, since

D_B and D_F are torsion-free. Moreover, $\dim E \leq \dim F + \dim B = r + n$, so $\dim E = r + n$. It follows that E is an abelian duality space of dimension $r + n$.

To prove (2), suppose now F and E are abelian duality spaces. The same argument as above shows

$$H^{p+n}(E, \mathbb{Z}[G^{\text{ab}}]) \cong H^p(B, \mathbb{Z}[Q^{\text{ab}}]) \otimes_{\mathbb{Z}} D_F,$$

for all p , so $H^r(B, \mathbb{Z}[Q^{\text{ab}}]) \otimes_{\mathbb{Z}} D_F \cong D_E$. Since D_F and D_E are torsion-free, it follows that $H^p(B, \mathbb{Z}[Q^{\text{ab}}])$ is torsion-free, nonzero, and concentrated in dimension $p = r$. By hypothesis, $\dim B = r$, so B is an abelian duality space.

To prove (3), suppose E and B are abelian duality spaces. From (25), since B is of finite-type, we obtain

$$E_2^{r,q} \cong D_B \otimes H^q(F, \mathbb{Z}[N^{\text{ab}}]),$$

and $E_2^{p,q} = 0$ for $p \neq r$. So the spectral sequence degenerates again to give

$$H^{n+q}(E, \mathbb{Z}[G^{\text{ab}}]) \cong D_B \otimes_{\mathbb{Z}} H^q(F, \mathbb{Z}[N^{\text{ab}}]),$$

and the argument concludes as in case (2). \square

An analogous statement holds for groups in place of spaces: we simply replace the hypothesis “finite-type CW-complex of dimension n ” with “FP group of cohomological dimension n .”

4.6. Discussion and examples. We say a group G is an almost-direct product of free groups, if it can be expressed as an iterated semidirect product of the form $G = F_{n_d} \rtimes \cdots \rtimes F_{n_1}$, where each group F_{n_p} is free (of rank $n_p \geq 1$), and acts on each group F_{n_q} with $q > p$ via an automorphism inducing the identity in homology.

Proposition 4.13, part (1) has an immediate application.

Corollary 4.14. *If $G = F_{n_d} \rtimes \cdots \rtimes F_{n_1}$ is an almost-direct product of free groups, then G is an abelian duality group of dimension d .*

Example 4.15. Both the pure braid group P_n and the group of pure symmetric automorphisms $P\Sigma_n^+$ are almost direct products of the form $F_{n-1} \rtimes \cdots \rtimes F_1$. Thus they are abelian duality groups of dimension $n - 1$. \diamond

The two conditions from Definition 4.11 are necessary for Proposition 4.13 to hold.

Example 4.16. Let G be the (discrete) Heisenberg group, which we saw in Example 4.10 was not an abelian duality group. Since G is a split extension of \mathbb{Z} by \mathbb{Z}^2 , with monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that condition (1) is necessary. Moreover, G is also a central extension of \mathbb{Z}^2 by \mathbb{Z} , yet the map $d: H_2(\mathbb{Z}^2, \mathbb{Z}) \rightarrow H_1(\mathbb{Z}, \mathbb{Z})$ is an isomorphism, thus showing that condition (2) is necessary. \diamond

5. ABELIAN DUALITY, JUMP LOCI, AND PROPAGATION

In this section, we note some qualitative constraints that the abelian duality property imposes on a space. We begin by recalling the definitions of the Alexander invariants and jump loci associated with a space.

5.1. Alexander invariants. As usual, we let X be a connected, finite-type CW-complex, and $G = \pi_1(X, *)$ its fundamental group. Moreover, we let G^{ab} be the abelianization of G , and $R = \mathbb{Z}G^{\text{ab}}$.

Suppose $X^\nu \rightarrow X$ is a regular abelian cover, classified by an epimorphism $G \xrightarrow{\text{ab}} G^{\text{ab}} \xrightarrow{\nu} H$, where H is a (finitely generated) abelian group. By Shapiro's Lemma, $H_p(X^\nu, \mathbb{Z}) \cong H_p(X, \mathbb{Z}H)$, where $\mathbb{Z}H$ is viewed as an R -module via the map ν .

Proposition 5.1. *If X is an abelian duality space of dimension n , with dualizing module $D = H^n(X, \mathbb{Z}G^{\text{ab}})$, then*

$$(26) \quad H_*(X^\nu, \mathbb{Z}) \cong \text{Ext}_R^{n-*}(D, \mathbb{Z}H).$$

Proof. The claim follows at once from Proposition 4.8. \square

Consider now the universal abelian cover, $X^{\text{ab}} \rightarrow X$. By definition, the p -th Alexander invariant of X is the R -module $H_p(X^{\text{ab}}, \mathbb{Z}) = H_p(X, R)$.

Corollary 5.2. *Suppose X is an abelian duality space and G^{ab} is torsion-free. If the p -th Alexander invariant is non-zero, then its support has dimension at most $p + m - n$, where m is the rank of G^{ab} .*

Proof. By hypothesis, R is a domain, so $\text{Ext}_R^q(D, R)$ is supported in codimension q or higher, cf. [EHV92, Thm. 1.1]. On the other hand, the dimension of R equals m . The desired conclusion follows. \square

5.2. Jump loci of spaces. Let $\widehat{G} := \text{Hom}(G, \mathbb{k}^*)$ denote the group of multiplicative, \mathbb{k} -valued characters of G . We will identify $\widehat{G} = \widehat{G^{\text{ab}}}$ with $\text{m-Spec } R$.

Definition 5.3. The *characteristic varieties* of X are defined as

$$(27) \quad \mathcal{V}_{i,d}(X; \mathbb{k}) := \mathcal{V}_{i,d}(C_*(\widetilde{X}^{\text{ab}}, \mathbb{k})),$$

where $C_*(\widetilde{X}^{\text{ab}}, \mathbb{k})$ denotes the equivariant cellular chain complex of the universal abelian cover (with coefficients in \mathbb{k}), viewed as a chain complex over the group-ring $R = \mathbb{k}[G^{\text{ab}}]$.

The cohomological variant is defined analogously:

$$(28) \quad \mathcal{V}^{i,d}(X; \mathbb{k}) := \mathcal{V}^{i,d}(C^*(\widetilde{X}^{\text{ab}}, \mathbb{k})),$$

where $C^*(\widetilde{X}^{\text{ab}}, \mathbb{k}) = C_*(\widetilde{X}^{\text{ab}}, \mathbb{k})^\vee$ is the equivariant cochain complex of $\widetilde{X}^{\text{ab}}$.

We note that the support loci of the Alexander invariants of X are sometimes known as the Alexander varieties of X . We let

$$(29) \quad \begin{aligned} \mathcal{W}_{i,d}(X; \mathbb{k}) &:= \mathcal{W}_{i,d}(C_*(\widetilde{X}^{\text{ab}}, \mathbb{k})) \\ &= \text{supp} \bigwedge^d H_i(X, R). \end{aligned}$$

The varieties $\mathcal{W}^{i,d}(X; \mathbb{k})$ are defined analogously.

Remark 5.4. By Proposition 2.9, $\mathcal{V}^{i,d}(X; \mathbb{k}) = \iota(\mathcal{V}_{i,d}(X; \mathbb{k}))$, for all integers i and d , as noted in [KP14, Lem. 4.3]. On the other hand, we remark that there is no such tight relationship between $\mathcal{W}_i(X; \mathbb{k})$ and $\mathcal{W}^i(X; \mathbb{k})$; we will come back to this point in Example 6.3. \diamond

5.3. Propagation of characteristic varieties. Perhaps the most striking consequence of abelian duality is the following nestedness property of the characteristic varieties. Let \mathbb{k} be an algebraically closed field.

Theorem 5.5. *Suppose X is an abelian duality space of dimension n over \mathbb{k} . Then the characteristic varieties of X propagate: that is, for any character $\rho \in \widehat{G}$, if $H^p(X, \mathbb{k}_\rho) \neq 0$, then $H^q(X, \mathbb{k}_\rho) \neq 0$ for all $p \leq q \leq n$. Equivalently,*

$$\{\mathbb{1}\} = \mathcal{V}^0(X, \mathbb{k}) \subseteq \mathcal{V}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{V}^n(X, \mathbb{k}).$$

Proof. Since X is an abelian duality space, the equivariant cochain complex $C^*(\tilde{X}^{\text{ab}}, \mathbb{k})$ satisfies the hypotheses of Proposition 2.8. \square

By considering the trivial representation, we note in particular:

Corollary 5.6. *Suppose X is an abelian duality space of dimension $n \geq 1$. Then $b_p(X) > 0$, for all $0 \leq p \leq n$.*

Remark 5.7. In [Hm86], J.-C. Hausmann showed that an analogue of the Kan–Thurston Theorem holds for duality groups: given any finite CW-complex X , there is a duality group G and a map $BG \rightarrow X$ inducing an isomorphism in homology. In view of Corollary 5.6, no such result holds for abelian duality groups. \diamond

Example 5.8. Let S_g be a surface of genus $g > 1$. Let $\Pi_g = \pi_1(S_g)$. It is well-known that $S_g \simeq K(\Pi_g, 1)$, and Π_g is a 2-dimensional Poincaré duality group. That is, $H^p(S_g, \mathbb{Z}\Pi_g)$ is isomorphic to \mathbb{Z} if $p = 2$, and is 0 otherwise. One can check, though, that for all $\rho \in \text{Hom}(\Pi_g, \mathbb{k}^*)$ except the trivial representation, we have $H^1(S_g, \mathbb{k}_\rho) \cong \mathbb{k}^{2g-2}$ and $H^2(S_g, \mathbb{k}_\rho) = 0$. By Theorem 5.5, then, the surface of genus $g > 1$ is *not* an abelian duality space. This example also shows that propagation of characteristic varieties fails to hold, in general, for duality spaces. \diamond

We also note that the rank of G^{ab} must be at least the dimension of X . More precisely,

Proposition 5.9. *If X is an abelian duality space of dimension n over \mathbb{k} and $\text{char } \mathbb{k}$ does not divide the order of the torsion subgroup of $H_1(X, \mathbb{Z})$, then $\dim_{\mathbb{k}} H^1(X, \mathbb{k}) \geq n$.*

Proof. Set $m = \dim_{\mathbb{k}} H^1(X, \mathbb{k})$. Recall that $R = \mathbb{k}[G^{\text{ab}}]$, and the cochain complex of the universal abelian cover is a free resolution of the dualizing module D as an R -module. First suppose G^{ab} is torsion-free. Then $\text{pdim}_R D \leq n$. On the other hand, by Proposition 4.8, $\text{Tor}_n^R(D, \mathbb{k}) = H^0(X, \mathbb{k}) \neq 0$, where \mathbb{k} is the trivial module. By our hypothesis on $\text{char } \mathbb{k}$, R is a regular \mathbb{k} -algebra of (global) dimension m , from which it follows that $\text{pdim}_R D = n \leq m$. \square

Along the same lines, the dimensions of irreducible components of $\mathcal{V}^i(X, \mathbb{k})$ are not arbitrary. As the next proposition shows, such components correspond to associated primes of the dualizing module D .

Proposition 5.10. *Let X be an abelian duality space of dimension n . Suppose G^{ab} has rank m , and again that it has no torsion elements of order $\text{char } \mathbb{k}$. If $Z = V(P)$ is an irreducible component of $\mathcal{V}^i(X, \mathbb{k})$ for a prime ideal P , then $\text{pdim}_R R/P \geq n - i$. If, moreover, Z is Cohen–Macaulay, then $\dim Z \leq i + n - m$.*

Proof. As in the proof of [EPY03, Theorem 4.1(c)], we let Z be such a component of dimension d , and let ρ be the generic point of Z . Then $H^i(X, \kappa(\rho)) \neq 0$, so by Lemma 2.2 and Proposition 4.8, $\text{Tor}_{n-i}^R(D, R/P) \neq 0$. This implies $n - i \leq \text{pdim } R/P$. If Z is Cohen–Macaulay, then $\dim Z + \text{pdim}_R P = m$, from which the second claim follows. \square

Remark 5.11. Though we must impose a condition on the characteristic of \mathbb{k} if $H_1(X, \mathbb{Z})$ has a nontrivial torsion subgroup, we do not know of an example of an abelian duality space for which $H_1(X, \mathbb{Z})$ is *not* torsion-free. \diamond

5.4. Propagation of resonance varieties. In the previous section, we saw that the abelian duality property informed on the behavior of the characteristic varieties. Here, we note that the situation is analogous for the EPY property and resonance varieties.

First, we define the resonance varieties of a space. As before, let X be a connected, finite-type CW-complex. Consider the cohomology algebra $A = H^*(X, \mathbb{k})$. If $\text{char } \mathbb{k} = 2$, we will assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. In this case, it is readily checked that $a^2 = 0$ for every $a \in A^1$; thus, A can be viewed, in a natural way, as a module over the exterior algebra $E = \bigwedge A^1$.

Definition 5.12. The *resonance varieties* of X (over \mathbb{k}) are the resonance varieties of A , viewed as a module over E :

$$(30) \quad \mathcal{R}^{i,d}(X, \mathbb{k}) = \{a \in H^1(X, \mathbb{k}) \mid \dim_{\mathbb{k}} H^i(A, a \cdot) \geq d\}.$$

For simplicity, we shall write $\mathcal{R}^i(X, \mathbb{k}) = \mathcal{R}^{i,1}(X, \mathbb{k})$. For a group G with finite-type classifying space, we shall also write $\mathcal{R}^{i,d}(G, \mathbb{k}) = \mathcal{R}^{i,d}(K(G, 1), \mathbb{k})$.

We begin with the analogous result to Theorem 5.5, which is simply a restatement of Theorem 3.8, applied to the cohomology ring of a space.

Theorem 5.13. *Suppose X is a space of dimension n with the EPY property over a field \mathbb{k} . Then the resonance varieties of X propagate: that is, for any $a \in A^1$, if $H^p(A, a \cdot) \neq 0$, then $H^q(A, a \cdot) \neq 0$ for all $p \leq q \leq n$, where $A = H^*(X, \mathbb{k})$. Equivalently,*

$$\{0\} = \mathcal{R}^0(X, \mathbb{k}) \subseteq \mathcal{R}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{R}^n(X, \mathbb{k}).$$

We continue with a further examination of the interplay between the duality properties of a space and the nature of its resonance varieties. We start with a simple observation relating Poincaré duality to the top resonance variety.

Proposition 5.14. *Let M be a compact, connected, orientable manifold of dimension n . Then $\mathcal{R}^n(M, \mathbb{k}) = \{0\}$.*

Proof. Let $\mu \in H^n(M, \mathbb{Z}) \cong \mathbb{Z}$ be the generator defining the orientation on M , and $\mu_{\mathbb{k}}$ its image in $H^n(M, \mathbb{k}) \cong \mathbb{k}$. Given any $a \in H^1(M, \mathbb{k})$, Poincaré duality guarantees the existence of a cohomology class $b \in H^{n-1}(M, \mathbb{k})$ such that $a \cup b = \mu_{\mathbb{k}}$, and we are done. \square

The same argument proves the following: If G is a Poincaré duality group of dimension n , then $\mathcal{R}^n(G, \mathbb{k}) = 0$. In dimension $n = 3$, we can say a bit more.

Proposition 5.15 ([DS09], Prop. 5.1). *Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even, then $\mathcal{R}^1(M, \mathbb{C}) = H^1(M, \mathbb{C})$.*

Putting together Propositions 5.14 and 5.15, we obtain the following corollary.

Corollary 5.16. *Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do not propagate (in characteristic 0).*

We shall see a concrete instance of this phenomenon in Example 5.20 below.

5.5. Minimal complexes. Here, we find that, if X is a minimal CW-complex, then the abelian duality and EPY properties are related.

We say the CW-structure on X is *minimal* if the number of i -cells of X equals the Betti number $b_i(X)$, for every $i \geq 0$. Equivalently, the boundary maps in the cellular chain complex $C_*(X, \mathbb{Z})$ are all zero maps. In particular, the homology groups $H_i(X, \mathbb{Z})$ are all torsion-free, and thus resonance varieties are defined in all characteristics.

For instance, if M is a smooth, closed manifold admitting a perfect Morse function, then M has a minimal cell structure. Evidently, spheres, tori, and orientable surfaces, as well as products thereof are of this type.

Theorem 5.17 ([PS10], Thm. 12.6). *Let X be a minimal CW-complex. Then the linearization of the cochain complex $C^*(X^{\text{ab}}, \mathbb{k})$ coincides with the universal Aomoto complex of $A = H^*(X, \mathbb{k})$.*

To see how this works concretely, pick an isomorphism $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^m$, and identify the group ring $\mathbb{k}[\mathbb{Z}^m]$ with the Laurent polynomial ring $\Lambda = \mathbb{k}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Next, filter Λ by powers of the maximal ideal $I = (t_1 - 1, \dots, t_m - 1)$, and identify the associated graded ring, $\text{gr}(\Lambda)$, with the polynomial ring $S = \mathbb{k}[x_1, \dots, x_m]$, via the ring map $t_i - 1 \mapsto x_i$.

The minimality hypothesis allows us to identify $C_i(X^{\text{ab}}, \mathbb{k})$ with $\Lambda \otimes_{\mathbb{k}} H_i(X, \mathbb{k})$ and $C^i(X^{\text{ab}}, \mathbb{k})$ with $A^i \otimes_{\mathbb{k}} \Lambda$. Under these identifications, the boundary map $\partial_{i+1}^{\text{ab}}: C_{i+1}(X^{\text{ab}}, \mathbb{k}) \rightarrow C_i(X^{\text{ab}}, \mathbb{k})$ dualizes to a map $\delta^i: A^i \otimes_{\mathbb{k}} \Lambda \rightarrow A^{i+1} \otimes_{\mathbb{k}} \Lambda$. Let $\text{gr}(\delta^i): A^i \otimes_{\mathbb{k}} S \rightarrow A^{i+1} \otimes_{\mathbb{k}} S$ be the associated graded of δ^i , and let $\text{gr}(\delta^i)^{\text{lin}}$ be its linear part. Theorem 5.17 then provides an identification with $\mathbf{L}(A)$ from (11),

$$(31) \quad \text{gr}(\delta^i)^{\text{lin}} = d^i: A^i \otimes_{\mathbb{k}} S \longrightarrow A^{i+1} \otimes_{\mathbb{k}} S.$$

Theorem 5.18. *If X is a minimal CW-complex, and the E -module $A = H^*(X, \mathbb{k})$ has the EPY property, then the I -adic completion of $H^p(X, \mathbb{k}G^{\text{ab}})$ vanishes for $p \neq n$, where n is the socle degree of A .*

Proof. By our minimality assumption, the equivariant spectral sequence from [PS10] starts at

$$(32) \quad \begin{aligned} E_0^{pq} &= \text{gr}_p(C^{p+q}(X^{\text{ab}}, \mathbb{k})) \\ &\cong A^{p+q} \otimes S_p. \end{aligned}$$

Furthermore, the differential d_0 vanishes, and so $E_1^{pq} = E_0^{pq}$, with differential d_1 given by (31). Hence, $E_2^{pq} = H^{p+q}(A \otimes S, \delta)_p$. By our Koszulness assumption,

$$(33) \quad E_2^{pq} = \begin{cases} F(A)_p & \text{if } p+q = n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $d_2 = 0$, and so $E_{\infty}^{pq} = E_2^{pq}$. Finally, we also know from [PS10] that the spectral sequence converges to the I -adic completion of $H^*(X, \mathbb{k}H)$. The desired conclusion readily follows. \square

Question 5.19. In principle, there is no reason to expect that the conclusion of Theorem 5.18 would hold for the (uncompleted) modules $H^p(X, \mathbb{k}G^{\text{ab}})$. However, we have no example. If a minimal CW-complex X has the EPY property, must X be an abelian duality space?

5.6. Discussion and example. We conclude this section with several more examples. The first is a (non-formal) space for which the characteristic varieties propagate, but the resonance varieties do not.

Example 5.20. Once again, we consider the Heisenberg manifold of Example 4.10. It is readily checked that $\mathcal{V}^i(M) = \{1\}$ for all $i \leq 3$; thus, the characteristic varieties of M propagate. On the other hand, $H^1(M, \mathbb{Z}) = \mathbb{Z}^2$, and the cup product map $H^1 \otimes H^1 \rightarrow H^2$ vanishes. Therefore, resonance does not propagate: $\mathcal{R}^1(M, \mathbb{k}) = \mathbb{k}^2$, yet $\mathcal{R}^3(M, \mathbb{k}) = \{0\}$.

Finally, note that M is an S^1 -bundle over T^2 with Euler number 1, and thus admits a minimal cell decomposition. Hence, by Theorems 3.8 and 5.17, we see again that the manifold M is *not* an abelian duality space. This, despite the fact that M^{ab} is homotopy equivalent to S^1 , which of course is a PD_1 -space. \diamond

In the case of 2-complexes with non-negative Euler characteristic, propagation occurs because of dimensional considerations. However, such spaces need not be abelian duality spaces, as we see below.

Proposition 5.21. *Suppose X is a connected, finite 2-complex with $\chi(X) \geq 0$. If $H^1(X, \mathbb{k}) \neq 0$, the resonance and characteristic varieties of X both propagate.*

Proof. Clearly $\mathcal{V}^0(X) = \{1\}$, and $1 \in \mathcal{V}^i(X)$ for $i = 1, 2$, by hypothesis. Now consider a character $\rho \neq 1$. The universal cover of X has a cellular chain complex

$$C.(\tilde{X}) := \mathbb{k}[G]^{c_2} \longrightarrow \mathbb{k}[G]^{c_1} \longrightarrow \mathbb{k}[G]^{c_0},$$

for some integers c_i with $\chi(X) = c_2 - c_1 + c_0 \geq 0$. Then $H^*(X, \mathbb{k}_\rho)$ is computed by the complex

$$\text{Hom}_{\mathbb{k}[G]}(C.(\tilde{X}, \mathbb{k}_\rho)): \mathbb{k}^{c_2} \longleftarrow \mathbb{k}^{c_1} \longleftarrow \mathbb{k}^{c_0}.$$

Since $H^0(X, \mathbb{k}_\rho) = 0$, we have $c_2 - c_1 \geq 0$, which means if $\rho \in \mathcal{V}^1(X, \mathbb{k})$, then $\rho \in \mathcal{V}^2(X, \mathbb{k})$. The analogous argument applies for the resonance varieties as well. \square

Finally, here is an example of a group which isn't an abelian duality group, yet for which resonance does propagate.

Example 5.22. Let X be the presentation 2-complex for the 1-relator group

$$G = \langle x_1, x_2 \mid [x_1, x_2][x_2, [x_2, x_1]] = 1 \rangle.$$

Since the relator is not a proper power, X is a $K(G, 1)$. A standard computation shows that $H^*(X, \mathbb{Z}) = H^*(T^2, \mathbb{Z})$; in particular, G is a PD_2 -group of nonnegative Euler characteristic, so the proposition above applies to show that its resonance and characteristic varieties propagate. Indeed, $\mathcal{R}^1(X, \mathbb{k}) = \mathcal{R}^2(X, \mathbb{k}) = \{0\}$, and moreover,

$$\mathcal{V}^1(X, \mathbb{k}) = \mathcal{V}^2(X, \mathbb{k}) = \{(t_1, t_2) \in (\mathbb{k}^*)^2 \mid t_2 = 2\} \cup \{1\}.$$

Next we note that the Euler characteristic assumption is necessary. Let $Y = X \vee S^1$. Again, X is an aspherical, minimal 2-complex. Moreover, both $\mathcal{R}^1(Y, \mathbb{k})$ and $\mathcal{R}^2(Y, \mathbb{k})$ are equal to $H^1(Y, \mathbb{k}) = \mathbb{k}^3$, and so resonance propagates for Y . On the other hand, $\mathcal{V}^1(Y, \mathbb{k}) = H^1(Y, \mathbb{k}^*) = (\mathbb{k}^*)^3$, yet

$$\mathcal{V}^2(Y, \mathbb{k}) = \{(t_1, t_2, t_3) \in (\mathbb{k}^*)^3 \mid t_2 = 2\} \cup \{1\}.$$

Therefore, if $\text{char}(\mathbb{k}) \neq 2$, the characteristic varieties of Y do not propagate. By Theorem 5.5, the space Y is not an abelian duality space. \diamond

6. HYPERPLANE ARRANGEMENTS

In this section, we give some examples of abelian duality spaces.

6.1. Linear arrangements. Let \mathcal{A} be an essential, central hyperplane arrangement in \mathbb{C}^{n+1} . Let $M(\mathcal{A}) = \mathbb{C}^{n+1} \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of the arrangement, and $U(\mathcal{A}) = \mathbb{P}^n \setminus \bigcup_{H \in \mathcal{A}} \overline{H}$ the projective complement. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$, and $U(\mathcal{A})$ may be regarded as the complement of a possibly non-central arrangement in \mathbb{C}^{n-1} by a choice of a hyperplane at infinity. Without loss of generality, then, we mostly restrict our attention to projective complements. $U(\mathcal{A})$ is a Stein manifold, and thus has the homotopy type of a connected CW-complex of dimension n . Moreover, as shown in [DP03, Ra02], the cell structure can be chosen to be minimal.

The cohomology ring $H^*(U(\mathcal{A}), \mathbb{Z})$ was computed by Brieskorn in the early 1970s, building on pioneering work of Arnol'd on the cohomology ring of the braid arrangement. It follows from Brieskorn's work that the space $U(\mathcal{A})$ is formal. We let $A(\mathcal{A}) := H^*(U(\mathcal{A}), \mathbb{Z})$, the (projective) Orlik-Solomon algebra of \mathcal{A} .

As shown in [DLJO11], the complement $U(\mathcal{A})$ is a duality space. It turns out that a hyperplane complement is also an abelian duality space. Both of these properties follow from a more general cohomological vanishing result developed in [DSY16], an instance of which is the following:

Theorem 6.1 ([DSY16], Thm. 5.6). *Let $U(\mathcal{A})$ be the complement of an essential arrangement in \mathbb{P}^n , and set $H = H_1(U(\mathcal{A}), \mathbb{Z})$. Then $H^p(U(\mathcal{A}), \mathbb{Z}H) = 0$ for all $p \neq n$, and $D(\mathcal{A}) := H^n(U(\mathcal{A}), \mathbb{Z}H)$ is free abelian.*

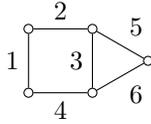
A main result of [EPY03] is that the shifted cohomology ring $A(\mathcal{A})(n)$ is a Koszul module over the exterior algebra $E = \bigwedge A^1(\mathcal{A})$: that is, it satisfies the EPY property. So arrangement complements satisfy the hypotheses of both Theorems 5.5 and 5.13.

Corollary 6.2. *Let \mathcal{A} be a hyperplane arrangement of rank $n+1$. Then both the characteristic and the resonance varieties of its projective complement, $U = U(\mathcal{A})$, propagate:*

$$\mathcal{V}^1(U, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{V}^n(U, \mathbb{k}) \quad \text{and} \quad \mathcal{R}^1(U, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{R}^n(U, \mathbb{k}).$$

6.2. Examples and discussion. We now give an example of an arrangement for which the support loci of the Orlik-Solomon algebra do *not* propagate.

Example 6.3. Let \mathcal{A} denote the graphic arrangement defined by equations $x_i - x_j$ for edges $\{i, j\}$ in the graph below.



If π is a partition of $[6]$, we let P_π denote the codimension- k linear subspace of V given by equations $\sum_{j \in \pi_s} x_j = 0$, for each $1 \leq s \leq k$, where $k = |\pi|$.

Let $A = H^*(U(\mathcal{A}), \mathbb{Q})$. Calculations as in [De16] show that $\mathcal{W}^1(\mathbf{L}(A)) = P_{1|2|3|4|5|6}$ and $\mathcal{W}^2(\mathbf{L}(A)) = P_{1234|5|6}$; we illustrate Proposition 3.10 by noting that

$$\begin{aligned} \mathcal{R}^0(A) &= \mathcal{W}_0(\mathbf{L}(A)) = P_{1|2|3|4|5|6}, \\ \mathcal{R}^1(A) &= \mathcal{W}_1(\mathbf{L}(A)), \\ \mathcal{R}^2(A) &= \mathcal{W}_1(\mathbf{L}(A)) \cup \mathcal{W}_2(\mathbf{L}(A)), \quad \text{and} \\ \mathcal{R}^3(A) &= \mathcal{W}_3(\mathbf{L}(A)) = P_{123456}. \end{aligned}$$

Since $\mathcal{W}^1(\mathbf{L}(A)) \not\subseteq \mathcal{W}^2(\mathbf{L}(A))$, we see in particular that, unlike the resonance varieties, the support loci of the complex $\mathbf{L}(A)$ do not have the propagation property. \diamond

Remark 6.4. In [Bu11, Bu14], Budur establishes the following inclusions for the complement M of an arrangement of rank n :

$$\begin{aligned} \mathcal{R}^i(M, \mathbb{C}) &\subseteq \mathcal{R}_2^i(M, \mathbb{C}) && \text{for } i \leq n - 2, \\ \mathcal{R}^i(M, \mathbb{C}) &\subseteq \mathcal{R}_d^i(M, \mathbb{C}) && \text{for } i < n - 2 \text{ and } d < 1 + (n - 3)/(i + 1). \end{aligned}$$

It would be interesting to see whether this type of ‘deeper’ propagation arises in the more general setup of minimal, abelian duality spaces. \diamond

6.3. Elliptic arrangements. Let E be an elliptic curve. An elliptic arrangement in $E^{\times n}$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_m\}$ of fibers of group homomorphisms $E^{\times n} \rightarrow E$, see [LV12, Bi16]. Such homomorphisms are parameterized by integer vectors: writing E as an additive group, we may write $H_i = f_i^{-1}(\zeta_i)$ for some point $\zeta_i \in E$, where

$$(34) \quad f_i(x_1, \dots, x_n) = \sum_{j=1}^n a_{ij}x_j,$$

and $A = (a_{ij})$ is a $m \times n$ integer matrix. We let $\text{corank}(\mathcal{A}) := n - \text{rank}(A)$.

Let $U(\mathcal{A}) = E^{\times n} \setminus \bigcup_{i=1}^m H_i$ be the complement of our elliptic arrangement. We show in [DSY16, Cor. 6.4] that $U(\mathcal{A})$ is both a duality and an abelian duality space of dimension $n + r$, where r is the corank of \mathcal{A} . Applying Theorem 5.5, we obtain the following immediate corollary.

Corollary 6.5. *Let \mathcal{A} be an elliptic arrangement in $E^{\times n}$. Then the characteristic varieties of its complement, $U = U(\mathcal{A})$, propagate:*

$$\mathcal{V}^1(U, \mathbb{k}) \subseteq \dots \subseteq \mathcal{V}^n(U, \mathbb{k}).$$

Remark 6.6. We do not know whether complements of elliptic arrangements satisfy the EPY property, or whether the resonance varieties of elliptic arrangements propagate. \diamond

6.4. Milnor fibers of linear arrangements. Now let $f_{\mathcal{A}}$ be a reduced defining polynomial for $\bigcup_{H \in \mathcal{A}} H$, a product of linear forms. The evaluation map $f_{\mathcal{A}}: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ restricts to a fibration, $F(\mathcal{A}) \rightarrow M(\mathcal{A}) \rightarrow \mathbb{C}^*$, known as the Milnor fibration. For a full discussion, see [DS14, Su14’]. The monodromy action of $\pi_1(\mathbb{C}^*)$ on $H_*(F, \mathbb{Z})$ is determined by the action of a generator, $h_*: H_*(F, \mathbb{Z}) \rightarrow H_*(F, \mathbb{Z})$. The order of h_* divides $N := \deg f_{\mathcal{A}} = |\mathcal{A}|$.

Theorem 6.7. *Suppose that \mathcal{A} is an essential, central arrangement of rank $n + 1$, and the monodromy action of h_1 on $H_1(F(\mathcal{A}), \mathbb{Z})$ is trivial. Then $F(\mathcal{A})$ is an abelian duality space of dimension n .*

Proof. We know $M(\mathcal{A})$ is an abelian duality space, by Theorem 6.1. By Remark 4.12, the fibration sequence $F(\mathcal{A}) \rightarrow M(\mathcal{A}) \rightarrow \mathbb{C}^*$ is ab-exact provided that h_1 is the identity map. The conclusion then follows from Proposition 4.13. \square

Remark 6.8. The condition that h_1 acts nontrivially on $H_1(F, \mathbb{Z})$ is interesting but rather special: see [Su14’, Thm. 5.1] for a complete discussion. For most arrangements (in some sense), then, the Milnor fiber is an abelian duality space.

Nevertheless, even if the monodromy action of h_1 on $H_1(F, \mathbb{Z})$ is non-trivial, the Milnor fiber F can still be an abelian duality space. For instance, if \mathcal{A} is a pencil of 3 lines in \mathbb{C}^2 , then the characteristic polynomial of h_1 is $(1 - t)(1 - t^3)$, yet $F(\mathcal{A}) = E \setminus \{1, \omega, \omega^2\}$, where $\omega^3 = 1$, and the claim follows from the discussion in §6.3. It would be interesting to know if the Milnor fiber of an arrangement is an abelian duality space in general.

We note that, h_p acts nontrivially on $H_p(F(\mathcal{A}), \mathbb{Z})$ for *some* p unless \mathcal{A} is a decomposable arrangement and the number of hyperplanes in each block have greatest common divisor 1: see [Di12]. We also note that other authors consider the (non)triviality of h_1 over \mathbb{C} rather than \mathbb{Z} ; however, we cannot rule out the possibility that $H_1(F, \mathbb{Z})$ has torsion: we refer to the discussion in [DS14]. \diamond

The propagation of characteristic varieties we observed for arrangement complements in Corollary 6.2 implies the following property of monodromy eigenspaces of Milnor fibers. If $\zeta \in \mathbb{k}^*$ is an eigenvalue of h on $H_*(F(\mathcal{A}), \mathbb{k})$ for an algebraically closed field \mathbb{k} of characteristic relatively prime to N , let $H_p(F(\mathcal{A}), \mathbb{k})_\zeta$ denote the corresponding eigenspace.

Theorem 6.9. *If \mathcal{A} is an arrangement of rank $n + 1$ and $H_p(F(\mathcal{A}), \mathbb{k})_\zeta \neq 0$ for some $p < n$, then $H_q(F(\mathcal{A}), \mathbb{k})_\zeta \neq 0$ as well, for all eigenvalues ζ , and all $p \leq q \leq n$.*

Proof. Let $\delta: \pi_1(U(\mathcal{A})) \rightarrow \mathbb{Z}/N\mathbb{Z}$ denote the classifying map of the cover $F(\mathcal{A}) \rightarrow U(\mathcal{A})$. If ζ is a primitive N th root of unity, let $\mathbb{k}_i = \widehat{\delta}(\zeta^i)$. Then we recall that

$$\begin{aligned} H_p(F(\mathcal{A}), \mathbb{k}) &= \bigoplus_{i=0}^{N-1} H_p(U(\mathcal{A}), \mathbb{k}_i) \\ &= \bigoplus_{i=0}^{N-1} H_p(F(\mathcal{A}), \mathbb{k})_{\zeta^i} \end{aligned}$$

for all p . The result then follows from Corollary 6.2. \square

7. RIGHT-ANGLED ARTIN GROUPS AND THEIR RELATIVES

7.1. Polyhedral products. We start by recalling the construction of a ‘generalized moment-angle complex’ (or, ‘polyhedral product’); for more details, we refer to [DS07, BCG10, Az13] and references therein.

Let $(\underline{X}, \underline{A}) := (X_i, A_i)_{1 \leq i \leq m}$ be an m -tuple of connected, finite CW-pairs, and let L be a simplicial complex on vertex set $\mathbf{V} = [m]$. We then set

$$(35) \quad \mathcal{Z}_L(\underline{X}, \underline{A}) = \bigcup_{\tau \in L} (\underline{X}, \underline{A})^\tau,$$

where $(\underline{X}, \underline{A})^\tau = \prod_{i \in \mathbf{V}} \underline{X}_{\tau, i}$, and

$$(36) \quad \underline{X}_{\tau, i} = \begin{cases} X_i & \text{if } i \in \tau; \\ A_i & \text{if not.} \end{cases}$$

Here, we will mainly be interested in the case when each $A_i = * = e^0$, a distinguished zero-cell in each X_i . If, moreover, each $X_i = X$, we write $\mathcal{Z}_L(X, *)$ in place of $\mathcal{Z}_L(\underline{X}, *)$.

Now let $S^1 = e^0 \cup e^1$ be the circle, endowed with the standard cell decomposition. The resulting polyhedral product, $T_L = \mathcal{Z}_L(S^1, e^0)$, is a subcomplex of the m -torus T^m . Thus, T_L comes endowed with a minimal cell decomposition. The cohomology ring $H^*(T_L, \mathbb{k})$ is the exterior Stanley–Reisner ring $\mathbb{k}\langle L \rangle$, with generators the duals v^* , and relations the monomials corresponding to the missing faces of L .

The fundamental group $G_L = \pi_1(T_L)$ is the *right-angled Artin group* determined by the graph $\Gamma = L^{(1)}$, with presentation consisting of a generator v for each vertex v in \mathbf{V} , and a commutator relation $vw = wv$ for each edge $\{v, w\}$ in Γ . A classifying space for the group

$G_\Gamma = G_L$ is the toric complex T_Δ , where $\Delta = \Delta_L$ is the flag complex of L , i.e., the maximal simplicial complex with 1-skeleton equal to the graph Γ .

Both the characteristic and resonance varieties of a toric complex were computed in [PS09]. To state those results, start by identifying $H_1(T_K, \mathbb{Z}) = \mathbb{Z}^m$, with generators indexed by the vertex set $V = [m]$. This allows us to identify $H^1(T_L, \mathbb{k})$ with the vector space $\mathbb{k}^V = \mathbb{k}^m$, and $H^1(T_L, \mathbb{k}^\times)$ with the algebraic torus $(\mathbb{k}^\times)^V = (\mathbb{k}^\times)^m$. For each subset $W \subseteq V$, let \mathbb{k}^W be the respective coordinate subspace, and let $(\mathbb{k}^\times)^W$ be the respective algebraic subtorus.

Theorem 7.1 ([PS09], Thms. 3.8 and 3.12). *With notation as above,*

$$\mathcal{V}_d^i(T_L) = \bigcup_W (\mathbb{k}^\times)^W \quad \text{and} \quad \mathcal{R}_d^i(T_L, \mathbb{k}) = \bigcup_W \mathbb{k}^W$$

where, in both cases, the union is taken over all subsets $W \subset V$ for which

$$\sum_{\sigma \in L_{V \setminus W}} \dim_{\mathbb{k}} \tilde{H}_{i-1-|\sigma|}(\text{lk}_{L_W}(\sigma), \mathbb{k}) \geq d.$$

In the above, L_W denotes the simplicial subcomplex induced by L on W , and $\text{lk}_K(\sigma)$ denotes the link of a simplex σ in a subcomplex $K \subseteq L$. In particular,

$$(37) \quad \mathcal{V}^1(G_\Gamma, \mathbb{k}) = \bigcup_W (\mathbb{k}^\times)^W \quad \text{and} \quad \mathcal{R}^1(G_\Gamma, \mathbb{k}) = \bigcup_W \mathbb{k}^W,$$

where the union is taken over all maximal subsets $W \subset V$ for which the induced graph Γ_W is disconnected.

7.2. Cohen–Macaulay complexes. Recall that a n -dimensional simplicial complex L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology $\tilde{H}^*(\text{lk}(\sigma), \mathbb{Z})$ is concentrated in degree $n - |\sigma|$ and is torsion-free. The analogous definition can be made over any coefficient ring \mathbb{k} . For a fixed \mathbb{k} , the Cohen–Macaulayness of L is a topological property: it depends only on the homeomorphism type of L . For $\sigma = \emptyset$, the condition means that $\tilde{H}^*(L, \mathbb{Z})$ is concentrated in degree n ; it also implies that L is pure, i.e., all its maximal simplices have dimension n .

The Cohen–Macaulay property turns out to be equivalent to duality and abelian duality, as we see in the next two results.

Theorem 7.2 ([BM01, JM05]). *A right-angled Artin group G_Γ is a duality group if and only if the flag complex Δ_Γ is Cohen–Macaulay. Moreover, G_Γ is a Poincaré duality group if and only if Γ is a complete graph.*

Theorem 7.3. *Let L be a n -dimensional complex. Then L is Cohen–Macaulay over \mathbb{k} if and only if the toric complex T_L is an abelian duality space (of dimension $n + 1$).*

Corollary 7.4. *A right-angled Artin group G_Γ is an abelian duality group if and only if the flag complex Δ_Γ is Cohen–Macaulay.*

Proof of Theorem 7.3. Let $A = \mathbb{k}[G_L^{\text{ab}}]$, with the natural action of G_L . Then A is a free module over $\mathbb{k}[G_\tau]$, for any τ , hence Cohen–Macaulay. So if L is a Cohen–Macaulay complex over \mathbb{k} , Theorem 7.4 of [DSY16] states that $H^p(T_L, A) = 0$ for all $p \neq n + 1$, and $H^{n+1}(T_L, A)$ is free, so T_L is an abelian duality space.

To prove the converse, we recall the proof of that result uses a spectral sequence with E_2 term given by

$$(38) \quad E_2^{pq} = \bigoplus_{\tau \in L: q=2|\tau|} A_{G_\tau} \otimes_{\mathbb{k}} \tilde{H}^{p+|\tau|-1}(\text{lk}_L(\tau), \mathbb{k}).$$

This is a spectral sequence of A -modules, and we note that A_{G_τ} has codimension $|\tau|$, since it is a quotient of A by a regular sequence of length $|\tau|$. In particular, A_{G_τ} is nonzero, so from (38), L is Cohen–Macaulay over \mathbb{k} if and only if $E_2^{pq} = 0$ whenever $p + q \neq n + 1$.

Now suppose that $H^p(T_L, A) = 0$ for all $p \neq n + 1$, so $E_\infty^{pq} = 0$ for $p + q \neq n + 1$. If L is not Cohen–Macaulay, then, choose the least integer q for which $E_2^{pq} \neq 0$ for some p where $p < n + 1 - q$. By our remarks above, E_2^{pq} is an A -module of codimension $q/2$. Since $E_\infty^{pq} = 0$, though, E_2^{pq} must be filtered by kernels of differentials d_r^{pq} for $r \geq 2$. The targets of such differentials all have strictly higher codimension as A -modules, though, which gives a contradiction. \square

The linearized version behaves in the same way.

Theorem 7.5. *The cohomology algebra $A = H^*(T_L, \mathbb{k})$ has the EPY property if and only if the complex L is Cohen–Macaulay over \mathbb{k} .*

Proof. Set $n = 1 + \dim L$. Let E be the exterior algebra over $H^1(T_L, \mathbb{k})$, let S be the symmetric algebra over $H^1(T_L, \mathbb{k})$, and let $J_L \subset S$ be the symmetric Stanley–Reisner ideal. By the construction of Aramova, Avramov and Herzog [AAH99], $A^*(n)$ has a linear free resolution (over E) for some n if and only if the $J_{L^*}(n)$ has a linear free resolution (over S), where L^* denotes the Alexander dual. By a result of Eagon and Reiner [ER98, Thm. 3], this is the case if and only if L is Cohen–Macaulay. \square

Remark 7.6. The Cohen–Macaulay property over a field \mathbb{k} depends on the characteristic. For example, if L is a flag triangulation of $\mathbb{R}\mathbb{P}^2$, then examining the link of the empty simplex shows L is not Cohen–Macaulay (integrally), though L is Cohen–Macaulay over any field \mathbb{k} of characteristic except 2. It follows from Theorem 7.3 that T_L is not an abelian duality space (integrally), though it is an abelian duality space over a field \mathbb{k} with $\text{char } \mathbb{k} \neq 2$.

Since T_L has dimension 3 and $H^1(T_L, \mathbb{Z})$ is torsion-free, we conclude that $H^2(T_L, \mathbb{Z}[G^{\text{ab}}])$ must be a nonzero abelian 2-group: direct calculation shows, in fact, that $H^2(T_L, \mathbb{k}[G^{\text{ab}}]) \cong \mathbb{k}[G^{\text{ab}}] \neq 0$ if $\text{char } \mathbb{k} = 2$. Analogously, the EPY property holds except in characteristic 2, a phenomenon observed first (in the symmetric Stanley–Reisner setting) in [ER98]. \diamond

7.3. Propagation. If the simplicial complex L is Cohen–Macaulay, then the characteristic and resonance varieties of the toric complex T_L (and the right-angled Artin group G_L) propagate, by Theorems 5.5 and 3.8. In view of Theorem 7.1, this has the following, purely combinatorial interpretation.

Corollary 7.7. *Let L be a Cohen–Macaulay complex over \mathbb{k} . Suppose there is a subset $W \subset V$ of the vertex set and a simplex σ supported on $V \setminus W$ such that $\tilde{H}_{i-1-|\sigma|}(\text{lk}_{L_W}(\sigma), \mathbb{k}) \neq 0$, for some $i \geq |\sigma|$. Then, for all $i \leq j \leq \dim(L) + 1$, there exists a subset $W \subset W' \subset V$ and a simplex σ' supported on $V \setminus W'$ such that $\tilde{H}_{j-1-|\sigma'|}(\text{lk}_{L_{W'}}(\sigma'), \mathbb{k}) \neq 0$.*

Question 7.8. Is there a direct, combinatorial proof of the result above?

As noted in [PS09, §3.5], the resonance varieties of toric complexes do not always propagate. For instance, if $\Gamma = \Gamma_1 \amalg \Gamma_2$, where $\Gamma_j = K_{m_j}$ are complete graphs on $m_j \geq 2$ vertices, $j = 1, 2$, and G_Γ is the corresponding right-angled Artin group, then $\mathcal{R}^1(G_\Gamma, \mathbb{k}) = \mathbb{k}^{m_1+m_2}$, yet $\mathcal{R}^i(G_\Gamma, \mathbb{k}) = \mathbb{k}^{m_1} \times \{0\} \cup \{0\} \times \mathbb{k}^{m_2}$ for $1 < i \leq \min(m_1, m_2)$.

Question 7.9. Is there a finite simplicial complex L which is not Cohen–Macaulay but for which the resonance varieties of T_L still propagate?

Remark 7.10. We saw in Proposition 3.7 that the Betti numbers of spaces with the EPY property satisfy certain inequalities, since they are the Betti numbers of a minimal linear resolution. By a result of Huh [Hu15], the Betti numbers of (linear) hyperplane arrangement complements are log-concave, that is, $b_{i-1}b_{i+1} \leq b_i^2$ for $1 \leq i \leq n-1$, which implies, in particular, that they form a unimodal sequence.

However, the example of toric complexes cautions us not to expect much in general. The Betti numbers of T_L are given by the f -vector of L , and an example of Björner [Bj81] shows that, even in the case of a simplicial sphere, the f -vector need not be unimodal. \diamond

7.4. More general polyhedral products. Here, we briefly consider groups that arise from more general parameters in the polyhedral product construction. That is, we let $X_i = \bigvee_{\tau_i} S^1$ for $1 \leq i \leq m$, and consider the space $\mathcal{Z}_L(\underline{X}, *)$. This is an instance of the following operadic construction for polyhedral products, first considered by Ayzenberg in [Az13].

Definition 7.11. Suppose L is a simplicial complex on the vertex set $[m]$, for some $n \geq 1$, and K_1, \dots, K_m are simplicial complexes on disjoint vertex sets V_1, \dots, V_m , respectively. We define the composition $L \circ (K_1, \dots, K_m)$ to be the simplicial complex with vertices $V := \bigcup_{i=1}^m V_i$, whose simplices are all sets of the form

$$(39) \quad \bigcup_{\substack{i: i \in \tau, \\ \sigma_i \in K_i}} \sigma_i,$$

where $\tau \in L$. Equivalently, $L \circ (K_1, \dots, K_m)$ is the union of the simplicial joins $K_{i_1} \star \dots \star K_{i_k}$ over all simplices $\tau = \{i_1, \dots, i_k\}$ of L .

Example 7.12. If (j_1, \dots, j_m) are positive integers and Δ^k denotes the k -dimensional simplex, then $L^* \circ (\Delta^{j_1-1}, \dots, \Delta^{j_m-1}) = L(J)^*$, where $-^*$ here denotes Alexander duality, and $L(J)$ is the simplicial wedge construction introduced in [BBCG15]. \diamond

We will need the following, which is similar to a result of Ayzenberg [Az13, Prop. 5.1]. For the convenience of the reader, we supply a short, self-contained proof.

Proposition 7.13. *Suppose L and K_1, \dots, K_n are simplicial complexes as above. Suppose (X_{ij}, A_{ij}) is a pair of (nonempty) finite CW-complexes, for each $1 \leq i \leq n$ and $j \in V_i$. Let $\underline{X} := (X_{ij})_{1 \leq i \leq m, j \in V_i}$ and $\underline{X}_i = (X_{ij})_{j \in V_i}$ for each $1 \leq i \leq m$. Define \underline{A} and \underline{A}_i similarly. For each i , let $Y_i = \mathcal{Z}_{K_i}(\underline{X}_i, \underline{A}_i)$, and $B_i = \prod_{j \in V_i} A_{ij}$. Then there is a natural homeomorphism of CW-complexes,*

$$\mathcal{Z}_L(\underline{Y}, \underline{B}) \cong \mathcal{Z}_{L \circ (K_1, \dots, K_m)}(\underline{X}, \underline{A}).$$

Proof. Both spaces are subcomplexes of $\prod_{i=1}^m \prod_{j \in V_j} X_{ij}$, constructed as unions of products. To see they are unions of the same products, consider a simplex S of $L \circ (K_1, \dots, K_m)$. By construction, $S = \bigcup_{i \in \tau} \sigma_i$ for some $\tau \in L$ and choice of $\sigma_i \in K_i$, for each $i \in \tau$.

By definition, $(\underline{X}, \underline{A})^S = \prod_{j \in V} \underline{X}_{S,j}$, where

$$(40) \quad \underline{X}_{S,j} = \begin{cases} X_{ij} & \text{if } j \in \sigma_i \text{ and } i \in \tau; \\ A_{ij} & \text{if not.} \end{cases}$$

(In the first case, we recall that i is (uniquely) determined by j , since $V = \coprod_i V_i$.)

On the other hand, $(\underline{Y}, \underline{B})^\tau = \prod_{i=1}^m \underline{Y}_{\tau,i}$, where

$$(41) \quad \underline{Y}_{\tau,i} = \begin{cases} \mathcal{Z}_{K_i}(\underline{X}_i, \underline{A}_i) & \text{if } i \in \tau, \text{ and} \\ B_i & \text{if not.} \end{cases}$$

Note that $\underline{Y}_{\tau,i} \subseteq \prod_{j \in V_i} X_{ij}$, for each i . Comparing its j th coordinate with (40), we see the two are equal. \square

7.5. Reinterpreting Cohen–Macaulayness. Now we return to the goal of this section. Given a simplicial complex L with m vertices, let K_i be the zero-dimensional complex with r_i vertices. Let $X_i = \mathcal{Z}_{K_i}(S^1, *) \cong \bigvee_{r_i} S^1$. Then Proposition 7.13 shows that $\mathcal{Z}_L(\underline{X}, *)$ is homeomorphic to a toric complex on $N = \sum_{i=1}^m r_i$ vertices,

$$(42) \quad \mathcal{Z}_L(\underline{X}, *) \cong \mathcal{Z}_{L \circ (K_1, \dots, K_m)}(S^1, *).$$

The simplicial complex $L(r_1, \dots, r_m) := L \circ (K_1, \dots, K_m)$ is obtained from L by letting V_i be (disjoint) sets of r_i vertices for each i , and declaring S to be a simplex of \tilde{L} if and only if $\{i: V_i \cap S \neq \emptyset\}$ is a simplex of L of the same dimension as S .

We obtain the following generalization of Theorem 7.3.

Theorem 7.14. *The polyhedral product space $\mathcal{Z}_L(\underline{X}, *)$, where $X_i = \bigvee_{r_i} S^1$ for $1 \leq i \leq m$ and some positive integers r_1, \dots, r_m , is an abelian duality space if and only if L is Cohen–Macaulay.*

Proof. Following the approach for toric complexes, we cover $\mathcal{Z}_L(\underline{X}, *)$ with the sets $(\underline{X}, *)^\tau$, for each $\tau \in L$. The inclusion of $(\underline{X}, *)^\tau \in \mathcal{Z}_L(\underline{X}, *)$ is easily seen to be a retract, so $G_\tau := \pi_1((\underline{X}, *)^\tau, *)$ is a subgroup of $G := \pi_1(\mathcal{Z}_L(\underline{X}, *), *)$. Each G_τ is a product of free groups, hence an abelian duality group of dimension $|\tau|$. If A is a $\mathbb{Z}[G]$ -module we obtain a spectral sequence as in [DSY16] with

$$(43) \quad E_2^{pq} = \tilde{H}^{p+|\tau|-1}(\mathrm{lk}_L(\tau), H^{q-|\tau|}(G_\tau, A)).$$

Since $A = \mathbb{Z}[G^{\mathrm{ab}}]$ is a free $\mathbb{Z}[G]$ -module, $E_2^{pq} = 0$ unless $q = 2|\tau|$. If L is Cohen–Macaulay of dimension n , $E_2^{pq} = 0$ unless $p + |\tau| - 1 = n - |\tau|$. Combining, we see the spectral sequence degenerates at E_2 and gives the desired conclusion.

The converse is proven by the same argument as Theorem 7.3. \square

7.6. Bestvina–Brady groups. Similar questions can be asked about the Bestvina–Brady groups

$$(44) \quad N_\Gamma = \ker(\nu: G_\Gamma \rightarrow \mathbb{Z}),$$

where ν is the diagonal character, taking each standard generators $v \in G_\Gamma$ to $1 \in \mathbb{Z}$. We may view these groups as fundamental groups of \mathbb{Z} -covers of toric complexes which are directly analogous to the Milnor fibration, viewed as a \mathbb{Z} -cover, in the previous section.

Once again, we find that the covers share the duality properties of the base, subject to some necessary additional hypothesis. As shown by Bestvina and Brady in [BB97], the group N_Γ is of type FP if and only if the flag complex Δ_Γ is acyclic, in which case its cohomological dimension is the dimension of Δ_Γ . With this in mind, (classical) duality is characterized by the following recent result of Davis and Okun.

Proposition 7.15 ([DO12], Prop. 9.3). *Suppose Δ_Γ is \mathbb{k} -acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay over \mathbb{k} .*

Theorem 7.16. *Let Γ be a graph, and Δ_Γ its flag complex. Then N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen–Macaulay.*

Proof. Suppose Δ_Γ is an acyclic flag complex. By [BB97], N_Γ is of type FP of dimension $n = \dim \Delta_\Gamma$. In particular, Δ_Γ is connected, so, by [PS07, Lem. 3.2], the sequence

$$(45) \quad 1 \longrightarrow N_\Gamma \longrightarrow G_\Gamma \longrightarrow \mathbb{Z} \longrightarrow 0$$

is ab-exact, in the sense of Definition 4.11. Now further suppose that Δ_Γ is also Cohen–Macaulay. Then, by Proposition 7.2, G_Γ is an abelian duality group. Hence, by Proposition 4.13(3), N_Γ is also an abelian duality group.

Conversely, suppose N_Γ is an abelian duality group. Then N_Γ is of type FP, and so Δ_Γ must be acyclic. By Proposition 4.13(1), we see that G_Γ is also an abelian duality group, whence Δ_Γ is Cohen–Macaulay, by Theorem 7.3. \square

Once again, we find that the EPY property follows in step with abelian duality.

Theorem 7.17. *If Δ_Γ is \mathbb{k} -acyclic and Cohen–Macaulay over \mathbb{k} , then $H^*(N_\Gamma, \mathbb{k})$ has the EPY property.*

Proof. Let $L = \Delta_\Gamma$, so that $G_\Gamma = \pi_1(T_L)$, and let $N_\Gamma = \pi_1(\widetilde{T}_L)$, the \mathbb{Z} -cover of the toric complex classified by $a = \sum_{i=1}^n e_i \in H^1(T_L, \mathbb{Z})$. From Theorem 7.1, we note that $a \notin \mathcal{R}^1(T_L)$.

Recall that $A = H^*(T_L, \mathbb{k}) \cong E/J_L$, an exterior Stanley–Reisner ring. Since, by assumption, $\widetilde{H}^*(L, \mathbb{k}) = 0$, we may apply [PS09, Cor. 7.2] to conclude that the cohomology ring $B := H^*(\widetilde{T}_L, \mathbb{k})$ is isomorphic to $E/(J_L + (a))$. Multiplication by a gives a surjective homomorphism of graded E -modules, $E/(J_L + (a))(1) \rightarrow (J_L + (a))/J_L$; because a is nonresonant, this homomorphism is an isomorphism. We thus obtain an exact sequence of E -modules,

$$(46) \quad 0 \longrightarrow B(1) \longrightarrow A \longrightarrow B \longrightarrow 0,$$

hence also an exact sequence

$$(47) \quad 0 \longrightarrow B^*(n+1) \longrightarrow A^*(n+1) \longrightarrow B^*(n) \longrightarrow 0,$$

where $n = \dim L$, by taking duals and shifting.

By hypothesis, $A^*(n+1)$ is a Koszul module, which is to say $\mathrm{Tor}_p^E(A^*(n+1), \mathbb{k})^q$ is concentrated in degree $q = p$. By examining the graded long exact sequence of $\mathrm{Tor}_\bullet^E(-, \mathbb{k})$, we see by induction that $B^*(n)$ is also a Koszul E -module. A change-of-rings argument shows that $B^*(n)$ is Koszul as an $E/(a)$ -module as well, which completes the proof. \square

REFERENCES

- [AAH99] Annetta Aramova, Luchezar L. Avramov, and Jürgen Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 579–594. [MR1603874](#)2000c:13021 [7.2](#)
- [Az13] Anton Ayzenberg, *Composition of simplicial complexes, polytopes and multigraded Betti numbers*, Trans. Moscow Math. Soc. **74** (2013), no. 2, 175–202. [MR32235795](#) [7.1](#), [7.4](#), [7.4](#)
- [BBCG10] Anthony Bahri, Martin Bendersky, Frederick R. Cohen, and Samuel Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. **225** (2010), no. 3, 1634–1668. [MR2673742](#) (2012b:13053) [1.5](#), [7.1](#)
- [BBCG15] Anthony Bahri, Martin Bendersky, Frederick R. Cohen, and Samuel Gitler, *Operations on polyhedral products and a new topological construction of infinite families of toric manifolds*, Homology Homotopy Appl. **17** (2015), no. 2, 137–160. [MR3426378](#) [7.12](#)
- [BB97] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. [MR1465330](#) (98i:20039) [7.6](#), [7.6](#)
- [Bi16] Christin Bibby, *Cohomology of abelian arrangements*, Proc. Amer. Math. Soc. **144** (2016), no. 7, 3093–3104. [MR3487239](#) [6.3](#)

- [BE73] Robert Bieri and Beno Eckmann, *Groups with homological duality generalizing Poincaré duality*, Invent. Math. **20** (1973), 103–124. [MR0340449 \(49 #5204\)](#) [1.1](#), [4.1](#), [4.1](#), [4.3](#), [4.5](#)
- [Bj81] Anders Björner, *The unimodality conjecture for convex polytopes*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 2, 187–188. [MR598684 \(82b:52013\)](#) [7.10](#)
- [Bo84] Amand Borel et al., *Intersection cohomology*, Reprint of the 1984 edition, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. [MR2401086 \(2009k:14046\)](#)
- [BD13] Mitya Boyarchenko and Vladimir Drinfeld, *A duality formalism in the spirit of Grothendieck and Verdier*, Quantum Topol. **4** (2013), no. 4, 447–489. [MR3134025](#) [4.2](#)
- [BM01] Noel Brady and John Meier, *Connectivity at infinity for right angled Artin groups*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 117–132. [MR1675166 \(2001b:20068\)](#) [1.5](#), [7.2](#)
- [Br82] Kenneth S. Brown, *Cohomology of groups*, Grad. Texts in Math., vol. 87, Springer-Verlag, New York-Berlin, 1982. [MR0672956 \(83k:20002\)](#) [4.1](#), [4.2](#)
- [Bu11] Nero Budur, *Complements and higher resonance varieties of hyperplane arrangements*, Math. Res. Lett. **18** (2011), no. 5, 859–873. [MR2875860](#) [6.4](#)
- [Bu14] Nero Budur, *Erratum: Complements and higher resonance varieties of hyperplane arrangements*, Math. Res. Lett. **21** (2014), no. 1, 83–84. [MR3247040](#) [6.4](#)
- [BW15] Nero Budur and Botong Wang, *Cohomology jump loci of differential graded Lie algebras*, Compos. Math. **151** (2015), no. 8, 1499–1528. [MR3383165](#) [2.3](#)
- [DLJO11] Michael W. Davis, Ian Leary, Tadeusz Januszkiewicz, and Boris Okun, *Cohomology of hyperplane complements with group ring coefficients*, Int. Math. Res. Not. IMRN **2011**, no. 9, 2110–2116. [MR2806559](#) [4.3](#), [6.1](#)
- [DO12] Michael W. Davis and Boris Okun, *Cohomology computations for Artin groups, Bestvina-Brady groups, and graph products*, Groups Geom. Dyn. **6** (2012), no. 3, 485–531. [MR2961283](#) [1.5](#), [7.15](#)
- [De16] Graham Denham, *Higher resonance varieties of matroids*, in: Configurations Spaces, 41–66, Springer INdAM series, vol. 14, Springer, Cham, 2016. [6.3](#)
- [DS07] Graham Denham and Alexander I. Suciú, *Moment-angle complexes, monomial ideals, and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, 25–60. [MR2330154 \(2008g:55028\)](#) [1.5](#), [7.1](#)
- [DS14] Graham Denham and Alexander I. Suciú, *Multinets, parallel connections, and Milnor fibrations of arrangements*, Proc. London Math. Soc. **108** (2014), no. 6, 1435–1470. [MR3218315](#) [1.2](#), [6.4](#), [6.8](#)
- [DSY16] Graham Denham, Alexander I. Suciú, and Sergey Yuzvinsky, *Combinatorial covers and vanishing cohomology*, Selecta Mathematica **22** (2016), no. 2, 561–594. [MR3477330](#) [1.4](#), [6.1](#), [6.1](#), [6.3](#), [7.2](#), [7.5](#)
- [Dim04] Alexandru Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004. [MR2050072 \(2005j:55002\)](#) [4.2](#)
- [Di12] Alexandru Dimca, *Tate properties, polynomial-count varieties, and monodromy of hyperplane arrangements*, Nagoya Math. J. **206** (2012), 75–97. [MR2926485](#) [6.8](#)
- [DP03] Alexandru Dimca and Stefan Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. of Math. **158** (2003), no. 2, 473–507. [MR2018927 \(2005a:32028\)](#) [6.1](#)
- [DS09] Alexandru Dimca and Alexander I. Suciú, *Which 3-manifold groups are Kähler groups?*, J. Eur. Math. Soc. **11** (2009), no. 3, 521–528. [MR2505439 \(2011f:32041\)](#) [5.15](#)
- [ER98] John Eagon and Victor Reiner, *Resolutions of Stanley–Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275. [MR1633767 \(99h:13017\)](#) [7.2](#), [7.6](#)
- [Eis95] David Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995. [MR1322960 \(97a:13001\)](#) [2.1](#), [2.1](#)
- [Eis05] David Eisenbud, *The geometry of syzygies*, Grad. Texts in Math., vol. 229, Springer-Verlag, New York, 2005. [MR2103875 \(2005h:13021\)](#) [3.1](#), [3.1](#), [3.3](#)
- [HK84] J. Herzog and M. Kühl, *On the Betti numbers of finite pure and linear resolutions*, Comm. Algebra **12** (1984), no. 13–14, 1627–1646. [MR743307 \(85e:13021\)](#) [3.3](#)
- [EHV92] David Eisenbud, Craig Huneke, and Wolmer Vasconcelos, *Direct methods for primary decomposition*, Invent. Math. **110** (1992), no. 2, 207–235. [MR1185582 \(93j:13032\)](#) [5.1](#)
- [EPY03] David Eisenbud, Sorin Popescu, and Sergey Yuzvinsky, *Hyperplane arrangement cohomology and monomials in the exterior algebra*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4365–4383. [MR1986506 \(2004g:52036\)](#) [1.3](#), [3.3](#), [5.3](#), [6.1](#)
- [Fa75] F. Thomas Farrell, *Poincaré duality and groups of type (FP)*, Comment. Math. Helv. **50** (1975), 187–195. [MR0382479 \(52 #3362\)](#) [4.1](#)

- [FS82] Roger Fenn and Dennis Sjerve, *Duality and cohomology for one-relator groups*, Pacific J. Math. **103** (1982), no. 2, 365–375. [MR0705236 \(85b:20070\)](#) [4.4](#)
- [Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966. [MR0222093](#) [4.2](#)
- [Ha77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg, 1977. [MR0463157 \(57 #3116\)](#) [2.3](#)
- [Hm86] Jean-Claude Hausmann, *Every finite complex has the homology of a duality group*, Math. Ann. **275** (1986), no. 2, 327–336. [MR0854015 \(88b:55010\)](#) [5.7](#)
- [Hu15] June Huh, *h -vectors of matroids and logarithmic concavity*, Adv. Math. **270** (2015), 49–59. [MR3286530](#) [7.10](#)
- [Ive86] Birger Iversen, *Cohomology of sheaves*, Universitext, Springer-Verlag, Berlin, 1986. [MR842190 \(87m:14013\)](#) [4.2](#), [4.3](#)
- [JM05] Craig Jensen and John Meier, *The cohomology of right-angled Artin groups with group ring coefficients*, Bull. London Math. Soc. **37** (2005), no. 5, 711–718. [MR2164833 \(2006m:20054\)](#) [1.5](#), [7.2](#)
- [KS90] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren Math. Wiss., vol. 292, Springer-Verlag, Berlin, 1990. [MR1074006 \(92a:58132\)](#) [4.2](#)
- [KP14] Toshitake Kohno and Andrei Pajitnov, *Novikov homology, jump loci and Massey products*, Cent. Eur. J. Math. **12** (2014), no. 9, 1285–1304. [MR3201322](#) [5.4](#)
- [LV12] Andrey Levin and Alexander Varchenko, *Cohomology of the complement to an elliptic arrangement*, in: *Configuration spaces: geometry, combinatorics and topology*, 373–388, CRM Series, vol. 14, Ed. Norm., Pisa, 2012. [MR3203648](#) [6.3](#)
- [PS07] Stefan Papadima and Alexander I. Suciú, *Algebraic invariants for Bestvina-Brady groups*, J. Lond. Math. Soc. **76** (2007), no. 2, 273–292. [MR2363416 \(2009j:20048\)](#) [7.6](#)
- [PS09] Stefan Papadima and Alexander I. Suciú, *Toric complexes and Artin kernels*, Adv. Math. **220** (2009), no. 2, 441–477. [MR2466422 \(2010h:57007\)](#) [7.1](#), [7.1](#), [7.3](#), [7.6](#)
- [PS10] Stefan Papadima and Alexander I. Suciú, *The spectral sequence of an equivariant chain complex and homology with local coefficients*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2685–2721. [MR2584616 \(2011b:55017\)](#) [5.17](#), [5.5](#), [5.5](#)
- [PS14] Stefan Papadima and Alexander I. Suciú, *Jump loci in the equivariant spectral sequence*, Math. Res. Lett. **21** (2014), no. 4, 863–883. [MR3275650](#) [2.6](#), [2.3](#), [2.7](#)
- [Pl80] Andrew Pletsch, *Profinite duality groups. I*, J. Pure Appl. Algebra **16** (1980), no. 1, 55–74. [MR054970481k:12027](#) [4.2](#)
- [Ra02] Richard Randell, *Morse theory, Milnor fibers and minimality of hyperplane arrangements*, Proc. Amer. Math. Soc. **130** (2002), no. 9, 2737–2743. [MR1900880 \(2003e:32048\)](#) [6.1](#)
- [Se62] Jean-Pierre Serre, *Cohomologie galoisienne*, Cours au Collège de France, 1962–1963. Seconde édition. With a contribution by Jean-Louis Verdier. Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin-Heidelberg-New York, 1962/1963. [MR0180551 \(31 #4785\)](#) [4.2](#)
- [Su14] Alexander I. Suciú, *Characteristic varieties and Betti numbers of free abelian covers*, Int. Math. Res. Not. IMRN **2014** (2014), no. 4, 1063–1124. [MR3168402](#) [1.2](#)
- [Su14'] Alexander I. Suciú, *Hyperplane arrangements and Milnor fibrations*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 2, 417–481. [MR3205599](#) [1.2](#), [6.4](#), [6.8](#)
- [Sw69] Moss E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969. [MR0252485 \(40 #5705\)](#) [2.2](#)

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