

# ALEXANDER POLYNOMIALS: ESSENTIAL VARIABLES AND MULTIPLICITIES

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ABSTRACT. We explore the codimension one strata in the degree-one cohomology jumping loci of a finitely generated group, through the prism of the multivariable Alexander polynomial. As an application, we give new criteria that must be satisfied by fundamental groups of smooth, quasi-projective complex varieties. These criteria establish precisely which fundamental groups of boundary manifolds of complex line arrangements are quasi-projective. We also give sharp upper bounds for the twisted Betti ranks of a group, in terms of multiplicities constructed from the Alexander polynomial. For Seifert links in homology 3-spheres, these bounds become equalities, and our formula shows explicitly how the Alexander polynomial determines all the characteristic varieties.

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Characteristic varieties.** Let  $G$  be a finitely generated group. Each character  $\rho: G \rightarrow \mathbb{C}^*$  in the complex algebraic group  $\mathbb{T}_G = \text{Hom}(G, \mathbb{C}^*)$  gives rise to a rank 1 local

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2000 *Mathematics Subject Classification.* Primary 14F35, 20F34. Secondary 14M12, 55N25, 57M05, 57M25.

*Key words and phrases.* characteristic varieties, Alexander polynomial, almost principal ideal, multiplicity, twisted Betti number, quasi-projective group, boundary manifold, Seifert link.

<sup>1</sup>Partially supported by the CEEEX Programme of the Romanian Ministry of Education and Research, contract 2-CEEx 06-11-20/2006.

<sup>2</sup>Partially supported by NSF grant DMS-0311142.

system on  $G$ , denoted by  $\mathbb{C}_\rho$ . The *characteristic varieties* of  $G$  are the jumping loci for homology with coefficients in such local systems:

$$\mathcal{V}_k(G) = \{\rho \in \mathbb{T}_G \mid \dim_{\mathbb{C}} H_1(G; \mathbb{C}_\rho) \geq k\}.$$

An alternate description is as follows. Let  $\mathbb{Z}G$  be the group ring of  $G$ , with augmentation ideal  $I_G$ , and let  $G_{\text{abf}}$  be the maximal torsion-free abelian quotient of  $G$ . Note that  $\mathbb{C}G_{\text{abf}}$  is the coordinate ring of identity component of the character torus,  $\mathbb{T}_G^0$ . Finally, let  $A_G = \mathbb{Z}G_{\text{abf}} \otimes_{\mathbb{Z}G} I_G$  be the Alexander module. Then, at least away from the origin,  $\mathcal{V}_k(G) \cap \mathbb{T}_G^0$  coincides with the subvariety of  $\mathbb{T}_G^0$  defined by  $\mathcal{E}_k(A_G)$ , the ideal of codimension  $k$  minors of a presentation matrix for  $A_G$ .

We focus in this paper on  $\mathcal{W}_1(G)$ , the union of all codimension-one irreducible components of  $\mathcal{V}_1(G)$ , which are contained in  $\mathbb{T}_G^0$ . It turns out that this variety is closely related to another, even more classical invariant.

**1.2. The Alexander polynomial.** The group ring  $\mathbb{Z}G_{\text{abf}}$  may be identified with the Laurent polynomial ring  $\Lambda_n = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ , where  $n = b_1(G)$ . Our other object of study is the (multivariable) *Alexander polynomial*,  $\Delta^G$ , defined as the greatest common divisor of all elements of  $\mathcal{E}_1(A_G)$ , up to units in  $\Lambda_n$ .

After reviewing these basic notions in §2, in a more general context, we start in §3 by making precise the relationship between the Alexander polynomial of  $G$  and the codimension-one stratum of  $\mathcal{V}_1(G)$ . For example, if  $b_1(G) > 1$  and  $\Delta^G \neq 0$ , then  $\mathcal{W}_1(G)$  is simply the hypersurface defined by  $\Delta^G$ ; for the remaining cases, see Corollary 3.2. As an application, we compute in Proposition 3.4 the Alexander polynomials of arrangement groups: if  $\mathcal{A}$  is a finite set of hyperplanes in some  $\mathbb{C}^\ell$ , and  $G$  is the fundamental group of its complement, then  $\Delta^G$  is constant, except when  $\mathcal{A}$  is a pencil, and  $|\mathcal{A}| \geq 3$ .

We then ask: When does the Alexander polynomial of  $G$  have a *single essential variable*, i.e.,

$$\Delta^G(t_1, \dots, t_n) \doteq P(t_1^{e_1} \cdots t_n^{e_n}),$$

for some polynomial  $P \in \mathbb{Z}[t^{\pm 1}]$ ? Equivalently, when is the Newton polytope of  $\Delta^G$  a line segment? If  $b_1(G) \geq 2$ , this happens precisely when  $\mathcal{W}_1(G) = \emptyset$ , or all the irreducible components of  $\mathcal{W}_1(G)$  are parallel, codimension-one subtori of  $\mathbb{T}_G^0$ ; see Proposition 3.7.

**1.3. Obstructions to quasi-projectivity.** Some 50 years ago, work of J.-P. Serre [21] raised the following problem: characterize those finitely presented groups that appear as the fundamental groups of smooth, connected, (quasi-) projective complex varieties. Such groups are called (*quasi-*) *projective groups*, see e.g. [1], [3].

A major goal in this note is to present and exploit new obstructions to quasi-projectivity of groups. Our approach in §4 is based on a foundational result of Arapura [2], which states: If  $W$  is an irreducible component of the first characteristic variety of a quasi-projective group  $G$ , then  $W = \rho \cdot \text{dir } W$ , with  $\rho \in \mathbb{T}_G$  and  $\text{dir } W \subset \mathbb{T}_G^0$  a subtorus through 1. Using refinements from [9] and [8], we formulate our first set of quasi-projectivity obstructions in terms of the relative position of the components of  $\mathcal{V}_1(G)$ , as follows (a more detailed statement is given in Theorem 4.2).

**Theorem A.** *Let  $G$  be a quasi-projective group. If  $W$  and  $W'$  are two distinct components of  $\mathcal{V}_1(G)$ , then either  $\text{dir } W = \text{dir } W'$ , or  $\text{dir } W \cap \text{dir } W'$  is a finite set.*

Using the dictionary between  $\mathcal{W}_1(G)$  and  $\Delta^G$ , we find that, with one exception, the Alexander polynomial of a quasi-projective group must have a single essential variable. More precisely, we prove the following result (a more detailed statement is given in Theorem 4.3).

**Theorem B.** *Let  $G = \pi_1(M)$  be the fundamental group of a smooth, connected, complex quasi-projective variety. If  $b_1(G) \neq 2$ , then the Alexander polynomial  $\Delta^G$  has a single essential variable. If  $M$  is projective, then  $\Delta^G \doteq \text{const}$ .*

The condition on the first Betti number of  $G$  is really necessary. In Example 4.6, we exhibit a smooth, non-compact algebraic surface  $M$ , with fundamental group  $G$ , for which  $b_1(G) = 2$  and the Newton polytope of  $\Delta^G$  is 2-dimensional.

As explained in Example 5.2, the characteristic varieties and Alexander polynomials of finitely presented groups can be very complicated. The restrictions imposed by our theorems are quite efficient when attacking Serre's problem. As an illustration, we consider groups of the form  $G_{\mathcal{A}} = \pi_1(M_{\mathcal{A}})$ , where  $\mathcal{A}$  is an arrangement of lines in the complex projective plane  $\mathbb{P}^2$ , and  $M_{\mathcal{A}}$  is the boundary of a regular neighborhood of  $\mathcal{A}$ . Using an explicit formula for  $\Delta^{G_{\mathcal{A}}}$ , recently obtained in [4], we determine in Proposition 4.7 the precise class of arrangements  $\mathcal{A}$  for which  $G_{\mathcal{A}}$  is a quasi-projective group.

**1.4. Twisted Betti ranks.** In Sections 5 and 6, we explore the connection between the multiplicities of the factors of the Alexander polynomial,  $\Delta^G$ , and the higher-depth characteristic varieties,  $\mathcal{V}_k(G)$ . We use this connection to give easily computable, sharp upper bounds for the 'twisted' Betti ranks,  $b_1(G, \rho) = \dim_{\mathbb{C}} H_1(G; \mathbb{C}_{\rho})$ , corresponding to non-trivial characters  $\rho \in \mathbb{T}_G$ .

Before stating our results, we need some more terminology. Given an irreducible Laurent polynomial  $f \in \mathbb{C}G_{\text{abf}}$ , define the 'generic' Betti number,  $b_1^{\text{gen}}(G, f)$ , as the largest integer  $k$  so that  $\mathcal{V}_k(G)$  contains  $V(f)$ . It turns out that  $b_1(G, \rho) = b_1^{\text{gen}}(G, f)$ , for  $\rho$  in a certain non-empty Zariski open subset of  $V(f)$ . Also, denote by  $\nu_{\rho}(f)$  the order of vanishing of the germ of  $f$  at  $\rho$ .

Finally, declare the Alexander ideal  $\mathcal{E}_1(A_G)$  to be *almost principal* if there is an integer  $d \geq 0$  such that  $I_{G_{\text{abf}}}^d \cdot (\Delta^G) \subset \mathcal{E}_1(A_G)$ , over  $\mathbb{C}$ . Examples abound: groups with  $b_1(G) = 1$ , link groups, as well as fundamental groups of closed, orientable 3-manifolds, see [17].

**Theorem C.** *Let  $\Delta^G \doteq \prod_{j=1}^s f_j^{\mu_j}$  be the decomposition into irreducible factors of the Alexander polynomial of a finitely generated group  $G$ . Then:*

- (i)  $b_1^{\text{gen}}(G, f_j) \leq \mu_j$ , for each  $j = 1, \dots, s$ .
- (ii) *If the Alexander ideal of  $G$  is almost principal, then, for all  $\rho \in \mathbb{T}_G^0 \setminus \{1\}$ ,*

$$b_1(G, \rho) \leq \sum_{j=1}^s \mu_j \cdot \nu_{\rho}(f_j).$$

This result is proved in Theorems 5.3 and 6.4. We also give a variety of examples showing that the inequalities can be strict, yet in general are sharp. When  $G_{\text{ab}}$  has no torsion and the upper bounds from Part (ii) above are attained, for every  $\rho$ , it follows that  $\Delta^G$  determines  $\mathcal{V}_k(G)$ , for all  $k$ .

In the setup from Theorem C(ii), suppose also that  $\Delta^G$  has a single essential variable. Then the ‘generic’ upper bounds from (i) take the form

$$(1) \quad b_1(G, \rho) \leq \mu_j, \text{ for all } \rho \in V(f_j) \setminus \{1\},$$

see Corollary 6.10. The simplest situation is when  $b_1(G) = 1$ , in which case the upper bounds (1) are all attained if and only if the monodromy of the complexified Alexander invariant is semisimple, see Proposition 6.13. For example, if  $G = \pi_1(M)$ , where  $M$  is a compact, connected manifold fibering over the circle with connected fibers, then  $b_1(G) = 1$  and the inequalities (1) are all equalities precisely when the algebraic monodromy is semisimple and with no eigenvalue equal to 1, see Corollary 6.15.

**1.5. Seifert links.** We conclude in Section 7 with a study of the characteristic varieties of a well known class of links in homology 3-spheres: those for which the link exterior admits a Seifert fibration. Since any such link,  $(\Sigma^3, L)$ , has an analytic representative given by an isolated, weighted homogeneous singularity (see [10], [18]), the corresponding link group,  $G_L = \pi_1(\Sigma^3 \setminus L)$ , is quasi-projective. In [10], Eisenbud and Neumann study in detail the Alexander-type invariants of Seifert link groups, obtaining a formula for the Alexander polynomial of  $G_L$ , in terms of a ‘splice diagram’ representing  $L$ .

The Eisenbud-Neumann calculus is used in Theorem 7.2 to show that the upper bounds from (1) all become equalities for Seifert links  $L$ . Consequently, the Alexander polynomial of  $L$  completely determines the characteristic varieties  $\mathcal{V}_k(G_L)$ , for all  $k \geq 1$ .

## 2. ALEXANDER INVARIANTS AND CHARACTERISTIC VARIETIES

We start by collecting some known facts about Alexander-type invariants and characteristic varieties, in a slightly more general context than the one outlined in the Introduction.

**2.1. Alexander polynomials.** Let  $R$  be a commutative ring with unit. Assume  $R$  is Noetherian and a unique factorization domain. If two elements  $\Delta, \Delta'$  in  $R$  generate the same principal ideal (that is,  $\Delta = u\Delta'$ , for some unit  $u \in R^*$ ), we write  $\Delta \doteq \Delta'$ .

Let  $A$  be a finitely-generated  $R$ -module. Then  $A$  admits a finite presentation,  $R^m \xrightarrow{M} R^n \rightarrow A \rightarrow 0$ . The  $i$ -th elementary ideal of  $A$ , denoted  $\mathcal{E}_i(A)$ , is the ideal in  $R$  generated by the minors of size  $n - i$  of the  $n \times m$  matrix  $M$ , with the convention that  $\mathcal{E}_i(A) = R$  if  $i \geq n$ , and  $\mathcal{E}_i(A) = 0$  if  $n - i > m$ . Clearly,  $\mathcal{E}_i(A) \subset \mathcal{E}_{i+1}(A)$ , for all  $i \geq 0$ . Of particular interest is the ideal of maximal minors,  $\mathcal{E}_0(A)$ , also known as the *order ideal*.

The  $i$ -th Alexander polynomial of  $A$ , denoted  $\Delta_i(A)$ , is a generator of the smallest principal ideal in  $R$  containing  $\mathcal{E}_i(A)$ , that is, the greatest common divisor of all elements of  $\mathcal{E}_i(A)$ . As such,  $\Delta_i(A)$  is well-defined only up to units in  $R$ . Note that  $\Delta_{i+1}(A)$  divides  $\Delta_i(A)$ , for all  $i \geq 0$ .

**Example 2.1.** Suppose  $A = R^s \oplus R/(\lambda_1) \oplus \cdots \oplus R/(\lambda_r)$ , where  $\lambda_1, \dots, \lambda_r$  are non-zero elements in  $R$  such that  $\lambda_r \mid \cdots \mid \lambda_1$ . Then

$$(2) \quad \Delta_i(A) \doteq \begin{cases} 0 & \text{if } 0 \leq i \leq s-1, \\ \lambda_{i-s+1} \cdots \lambda_r & \text{if } s \leq i \leq s+r-1, \\ 1 & \text{if } s+r \leq i. \end{cases}$$

This computation applies to any finitely generated module over a principal ideal domain  $R$ ; for instance,  $R = \mathbb{K}[t]$  with  $\mathbb{K}$  a field, or  $R$  a discrete valuation ring.

We will be mostly interested in the case when  $R$  is the ring of Laurent polynomials over the integers,  $\Lambda_q = \mathbb{Z}[t_1^{\pm 1}, \dots, t_q^{\pm 1}]$ , or over the complex numbers,  $\Lambda_q \otimes \mathbb{C} = \mathbb{C}[t_1^{\pm 1}, \dots, t_q^{\pm 1}]$ . We will denote by  $\doteq$  equality up to units in  $\Lambda_q$  (of the form  $u = \pm t_1^{\nu_1} \cdots t_q^{\nu_q}$ ), and by  $\doteq_{\mathbb{C}}$  equality up to units in  $\Lambda_q \otimes \mathbb{C}$  (of the form  $u = z t_1^{\nu_1} \cdots t_q^{\nu_q}$ , with  $z \in \mathbb{C}^*$ ). In particular, we will write  $\Delta \doteq \text{const}$  if  $\Delta \doteq c$ , for some  $c \in \mathbb{Z}$  (equivalently,  $\Delta \doteq_{\mathbb{C}} 0$  or  $1$ ).

Suppose  $A$  is a finitely-generated module over  $\Lambda_q$ . We then have Alexander polynomials,  $\Delta_i(A) \in \Lambda_q$  and  $\Delta_i(A \otimes \mathbb{C}) \in \Lambda_q \otimes \mathbb{C}$ , well defined up to units in the respective rings. Using the unique factorization property, it is readily seen that  $\Delta_i(A \otimes \mathbb{C}) \doteq_{\mathbb{C}} \Delta_i(A)$ .

**2.2. Support varieties.** As before, let  $A$  be a finitely-generated  $R$ -module. The *support* of  $A$  is the reduced subscheme of  $\text{Spec}(R)$  defined by the order ideal,  $\mathcal{E}_0(A)$ ; we will denote it by  $V_1(A)$ . Since  $\sqrt{\mathcal{E}_0(A)} = \sqrt{\text{ann } A}$ , this is the usual notion of support in algebraic geometry, based on the annihilator ideal of the module  $A$ . In particular, a prime ideal  $\mathfrak{p} \subset R$  belongs to  $V_1(A)$  if and only if the localized module  $A_{\mathfrak{p}}$  is non-zero.

More generally, the *k-th support variety* of  $A$ , denoted  $V_k(A)$ , is the reduced subscheme of  $\text{Spec}(R)$  defined by the ideal  $\mathcal{E}_{k-1}(A)$ . Clearly,  $\{V_k(A)\}_{k \geq 1}$  is a decreasing sequence of subvarieties of  $\text{Spec}(R)$ ; these varieties are invariants of the  $R$ -isomorphism type of  $A$ .

If  $\text{codim } V_k(A) > 1$ , then  $\Delta_{k-1}(A) \doteq 1$ , so  $\Delta_{k-1}(A)$  carries no useful information. One of our guiding principles in this note is to elaborate on this, by explaining how much can be extracted from the Alexander polynomials, in the case when they are non-constant.

**2.3. Alexander modules.** Let  $X$  be a connected CW-complex. Without loss of generality, we may assume that  $X$  has a unique 0-cell, call it  $x_0$ . We will make the further assumption that  $X$  has finitely many 1-cells, in which case the fundamental group,  $G = \pi_1(X, x_0)$ , is finitely generated.

Let  $\phi: G \twoheadrightarrow H$  be a homomorphism onto an abelian group  $H$ , and let  $p: X^\phi \rightarrow X$  be the corresponding Galois cover. Denote by  $F = p^{-1}(x_0)$  the fiber of  $p$  over the basepoint. The exact sequence of the pair  $(X^\phi, F)$  then yields an exact sequence of  $\mathbb{Z}H$ -modules,

$$(3) \quad 0 \longrightarrow H_1(X^\phi; \mathbb{Z}) \longrightarrow H_1(X^\phi, F; \mathbb{Z}) \longrightarrow H_0(F; \mathbb{Z}) \longrightarrow H_0(X^\phi; \mathbb{Z}) \longrightarrow 0.$$

The  $\mathbb{Z}H$ -modules  $B_\phi = H_1(X^\phi; \mathbb{Z})$  and  $A_\phi = H_1(X^\phi, F; \mathbb{Z})$  are called the *Alexander invariant* (respectively, the *Alexander module*) of  $X$ , relative to  $\phi$ . Identifying the kernel

of  $H_0(F; \mathbb{Z}) \rightarrow H_0(X^\phi; \mathbb{Z})$  with the augmentation ideal,  $I_H = \ker(\mathbb{Z}H \xrightarrow{\epsilon} \mathbb{Z})$ , yields the ‘Crowell exact sequence’ of  $X$ ,

$$(4) \quad 0 \longrightarrow B_\phi \longrightarrow A_\phi \longrightarrow I_H \longrightarrow 0.$$

By considering a classifying map  $X \rightarrow K(G, 1)$ , it is readily seen that both  $\mathbb{Z}H$ -modules,  $A_\phi$  and  $B_\phi$ , depend only on the homomorphism  $\phi$ . For example, if  $\phi_{\text{ab}}: G \rightarrow G_{\text{ab}}$  is the abelianization map (corresponding to the maximal abelian cover  $X^{\text{ab}} \rightarrow X$ ), then  $A_{\text{ab}} = \mathbb{Z}G_{\text{ab}} \otimes_{\mathbb{Z}G} I_G$  and  $B_{\text{ab}} = G'/G''$ , with  $\mathbb{Z}G_{\text{ab}}$ -module structure on  $B_{\text{ab}}$  determined by the extension

$$(5) \quad 0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G/G' \longrightarrow 0.$$

**2.4. Torsion-free abelian covers.** Assume now  $H$  is a finitely-generated, free abelian group; let  $q$  be its rank. Identify the group ring  $\mathbb{Z}H$  with  $\Lambda_q = \mathbb{Z}[t_1^{\pm 1}, \dots, t_q^{\pm 1}]$ , the augmentation ideal  $I_H$  with the ideal  $I = (t_1 - 1, \dots, t_q - 1)$ , and  $\mathbb{T}_H = \text{Spec}(\mathbb{C}H)$  with the algebraic torus  $\text{Hom}(H, \mathbb{C}^*) = (\mathbb{C}^*)^q$ . As before, let  $\phi: G \twoheadrightarrow H$  be an epimorphism.

**Proposition 2.2** ([5], [23]). *Let  $k$  be a positive integer. Then  $\mathcal{E}_{k-1}(B_\phi) \cdot I^{q-1} \subset \mathcal{E}_k(A_\phi)$ . Moreover, there is a positive integer  $r$  such that  $\mathcal{E}_k(A_\phi) \cdot I^{r+k-q} \subset \mathcal{E}_{k-1}(B_\phi)$ .*

**Corollary 2.3.** *For any integer  $k \geq 1$ , the subschemes of  $\mathbb{T}_H$  given by  $V(\mathcal{E}_{k-1}(B_\phi))$  and  $V(\mathcal{E}_k(A_\phi))$  coincide away from the identity element  $1 \in \mathbb{T}_H$ .*

The most important case for us is the projection onto the maximal torsion-free abelian quotient,  $\phi_{\text{abf}}: G \rightarrow G_{\text{abf}}$ , where  $G_{\text{abf}} = G_{\text{ab}}/\text{Torsion}$ . The resulting covering is the maximal torsion-free abelian cover of  $X$ , and the corresponding Alexander modules are the classical objects considered in the theory of knots and links; see e.g. [13], [17]. In the sequel, whenever  $\phi$  is not mentioned, we will assume that  $\phi$  is the morphism  $\phi_{\text{abf}}$ , and will denote the corresponding  $\mathbb{Z}G_{\text{abf}}$ -modules simply by  $B_G$  and  $A_G$ .

**2.5. Characteristic varieties.** As before, let  $X$  be a connected CW-complex with finite 1-skeleton, and  $G = \pi_1(X, x_0)$ . Consider the character group of  $G$ , i.e., the complex algebraic group

$$\mathbb{T}_G = \text{Hom}(G, \mathbb{C}^*) = \text{Hom}(G_{\text{ab}}, \mathbb{C}^*).$$

Its identity component,  $\mathbb{T}_G^0$ , may be identified with the complex torus  $\text{Hom}(G_{\text{abf}}, \mathbb{C}^*) = (\mathbb{C}^*)^n$ , where  $n = b_1(G)$ . Each character  $\rho: G \rightarrow \mathbb{C}^*$  gives rise to a rank 1 local system on  $X$ , denoted by  $\mathbb{C}_\rho$ . The *characteristic varieties* of  $X$  are the jumping loci for homology with coefficients in such local systems:

$$\mathcal{V}_k(X) = \{\rho \in \mathbb{T}_G \mid \dim_{\mathbb{C}} H_1(X; \mathbb{C}_\rho) \geq k\}.$$

It is readily seen that these sets form a decreasing sequence of algebraic subvarieties of  $\mathbb{T}_G$ , depending only on the isomorphism type of the group  $G$ . In particular, we may write  $\mathcal{V}_k(G) := \mathcal{V}_k(X)$ .

The Lemma below is well-known—see for example Hironaka [14]. For the reader’s convenience, we include a short proof, in a slightly more general context.

**Proposition 2.4.** *Let  $G$  be a finitely generated group. Then, for all  $k \geq 1$ ,*

$$\mathcal{V}_k(G) \cap \mathbb{T}_G^0 = V_{k+1}(A_G \otimes \mathbb{C}) \quad \text{and} \quad \mathcal{V}_k(G) = V_{k+1}(A_{\text{ab}} \otimes \mathbb{C})$$

*away from 1, with equality at 1 for  $k < b_1(G)$ .*

*Proof.* Let  $\rho \in \mathbb{T}_G^0 \setminus \{1\}$  be a non-trivial character. By the discussion in §2.3, we have

$$\dim_{\mathbb{C}} H_1(X; \mathbb{C}_\rho) \geq k \iff \dim_{\mathbb{C}} (\mathbb{C}_\rho \otimes_{\mathbb{C}G_{\text{abf}}} A_G \otimes \mathbb{C}) \geq k + 1.$$

This in turn is equivalent to  $\rho \in V_{k+1}(A_G \otimes \mathbb{C})$ , by the definitions from §§2.1 and 2.2.

It is readily checked that the identity belongs to  $\mathcal{V}_k(G)$  and to  $V_{k+1}(A_G \otimes \mathbb{C})$ , whenever  $k < b_1(G)$ . The proof for  $A_{\text{ab}}$  is similar.  $\square$

As a corollary, we obtain the following result (see also Delzant [6] for another proof).

**Corollary 2.5.** *The characteristic varieties  $\mathcal{V}_k(G)$  depend only on  $G/G''$ , the maximal metabelian quotient of  $G$ .*

*Proof.* Clearly, the projection  $G \twoheadrightarrow G/G''$  induces the identity on abelianizations. By (5), the  $\mathbb{Z}G_{\text{ab}}$ -module  $B_{\text{ab}} = G'/G''$  is determined by  $G/G''$ . By naturality of the Crowell exact sequence (4), the Alexander module  $A_{\text{ab}}$  also depends only on  $G/G''$ . The conclusion follows from Proposition 2.4.  $\square$

Let  $H$  be a finitely-generated, free abelian group, and  $\phi: G \twoheadrightarrow H$  an epimorphism. The induced map,  $\phi^*: \mathbb{T}_H \hookrightarrow \mathbb{T}_G$ , is an inclusion, mapping  $\mathbb{T}_H$  to a subtorus of  $\mathbb{T}_G^0$ . From Corollary 2.3 and Proposition 2.4, we obtain the following.

**Corollary 2.6.** *Let  $\rho$  be a non-trivial character in  $\mathbb{T}_H$ . For all  $k \geq 1$ ,*

$$\phi^*(\rho) \in \mathcal{V}_k(G) \iff \rho \in V(\mathcal{E}_{k-1}(B_\phi)).$$

**2.6. Finitely presented groups.** Recall  $X$  is a connected CW-complex with finite 1-skeleton, and so  $G = \pi_1(X)$  is finitely generated. For our purposes, we may assume without loss of generality that  $G$  is, in fact, finitely presented.

Indeed, we may construct a finitely presented group  $\bar{G}$ , together with an epimorphism,  $f: \bar{G} \twoheadrightarrow G$ , so that the induced map on abelianizations,  $f_*: \bar{G}_{\text{ab}} \rightarrow G_{\text{ab}}$ , is an isomorphism. Given a commutative ring  $S$ , a ring homomorphism  $\sigma: \mathbb{Z}G \rightarrow S$  puts a  $\mathbb{Z}G$ -module structure on  $S$ ; moreover, the homology groups of  $G$  with coefficients in this  $\mathbb{Z}G$ -module inherit an  $S$ -module structure. It is readily checked that the map  $\text{Hom}(f, \text{id}_S): \text{Hom}_{\text{Rings}}(\mathbb{Z}G, S) \rightarrow \text{Hom}_{\text{Rings}}(\mathbb{Z}\bar{G}, S)$  is an isomorphism. Thus,  $f_*: H_1(\bar{G}; S) \rightarrow H_1(G; S)$  is an  $S$ -isomorphism. It follows that the Alexander modules and the characteristic varieties of  $G$  and  $\bar{G}$  are isomorphic.

Now suppose  $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_h \rangle$  is a finite presentation. Let  $\partial r_i / \partial x_j$  be the Fox derivatives of the relators, viewed as elements in  $\mathbb{Z}G$ , and let  $\tilde{\phi}: \mathbb{Z}G \rightarrow \mathbb{Z}H$  be the linear extension of the homomorphism  $\phi: G \rightarrow H$ . As shown by R. Fox [12], the resulting matrix,

$$A_G^\phi = (\tilde{\phi}(\partial r_i / \partial x_j)): \mathbb{Z}H^h \rightarrow \mathbb{Z}H^m,$$

is a presentation matrix for the Alexander module  $A_\phi$ . In view of Proposition 2.4, this yields an algorithm for computing the characteristic varieties  $\mathcal{V}_k(G)$ , as the determinantal varieties of the matrix  $A_G^{\text{ab}}$ . The matrix  $A_G = A_G^{\text{abf}}$  will be called the *Alexander matrix* of  $G$ .

We shall have occasion to consider various base changes on the matrix  $(\partial r_i / \partial x_j)$ . If  $\psi: \mathbb{Z}G \rightarrow S$  is a ring homomorphism, then applying  $\psi$  to each entry yields a new matrix,  $A_G^\psi: S^h \rightarrow S^m$ .

### 3. CODIMENSION-ONE COMPONENTS OF CHARACTERISTIC VARIETIES

In this section, we analyze the extent to which the Alexander polynomial of a space  $X$  determines the codimension 1 irreducible components of the characteristic variety  $\mathcal{V}_1(X)$ .

**3.1. The codimension-one stratum of the support variety.** Let  $\mathcal{E}$  be a non-zero ideal in  $\Lambda_q \otimes \mathbb{C} = \mathbb{C}[t_1^{\pm 1}, \dots, t_q^{\pm 1}]$ , and set  $\Delta = \text{gcd}(\mathcal{E})$ . Denote by  $V(\Delta)$  and  $V(\mathcal{E})$  the corresponding zero sets in  $(\mathbb{C}^*)^q$ . Let  $W_1(\mathcal{E})$  be the union of all codimension-one irreducible components of  $V(\mathcal{E})$ .

**Lemma 3.1.** *With the above notation,  $W_1(\mathcal{E}) = V(\Delta)$ .*

*Proof.* We use the fact that  $\Lambda_q$  is a unique factorization domain. Let  $W$  be a codimension-1 component of  $V(\mathcal{E})$ . Then  $W = V(g)$ , for some irreducible Laurent polynomial  $g \in \Lambda_q$ . Since  $V(g) \subset V(\mathcal{E})$ , the polynomial  $g$  must divide  $\Delta$ , and so  $W \subset V(\Delta)$ .

Conversely, note that  $\Delta \neq 0$ , since  $\mathcal{E} \neq 0$ . If  $\Delta$  is a unit, then  $V(\Delta) = \emptyset$ . Otherwise, let  $\{f_j\}$  be the prime factors of  $\Delta$ , and  $V(\Delta) = \bigcup_j V(f_j)$  the irreducible decomposition. It follows that  $V(f_j) \subset V(\mathcal{E})$ , for all  $j$ . Since  $\dim V(\mathcal{E}) \leq n - 1$ , we infer that  $V(\Delta) \subset W_1(\mathcal{E})$ , and we are done.  $\square$

**3.2. Alexander polynomial.** Let  $X$  be a CW-complex with finite 1-skeleton, and let  $G = \pi_1(X, x_0)$  be its fundamental group. Suppose we are given a surjective homomorphism,  $\phi: G \rightarrow H$ , from  $G$  to a torsion-free, finitely generated abelian group  $H$ . Define the *Alexander polynomial* of  $X$  (relative to  $\phi$ ) as

$$\Delta_\phi^X := \Delta_1(A_\phi) \in \mathbb{Z}H.$$

From the discussion in §2, it follows that  $\Delta_\phi^X$  depends only on  $G$ , modulo units in  $\mathbb{Z}H$ ; consequently, we will write the Alexander polynomial as  $\Delta_\phi^G$ . In the case when  $H = G_{\text{abf}}$  and  $\phi = \phi_{\text{abf}}$ , we will simply write the Alexander polynomial as  $\Delta^G$ , and view it as an element in  $\Lambda_n$ , where  $n = b_1(G)$ .

Denote by  $V(\Delta^G)$  the hypersurface of  $\mathbb{T}_G^0 = (\mathbb{C}^*)^n$  defined by the Alexander polynomial of  $G$ . Furthermore, denote by  $W_1(G)$  the union of all codimension-one irreducible components of  $\mathcal{V}_1(G)$  contained in  $\mathbb{T}_G^0$ .

Using Proposition 2.4, Lemma 3.1, and the discussion at the end of §2.1, we obtain the following relationships between  $W_1(G)$  and  $\Delta^G$ .

**Corollary 3.2.** *Let  $G$  be a finitely generated group.*

- (1)  $\Delta^G = 0 \iff \mathbb{T}_G^0 \subset \mathcal{V}_1(G)$ . In this case,  $\mathcal{W}_1(G) = \emptyset$ .  
(2) If  $b_1(G) \geq 1$  and  $\Delta^G \neq 0$ , then

$$\mathcal{W}_1(G) = \begin{cases} V(\Delta^G) & \text{if } b_1(G) > 1 \\ V(\Delta^G) \amalg \{1\} & \text{if } b_1(G) = 1. \end{cases}$$

- (3) If  $b_1(G) \geq 2$ , then

$$\mathcal{W}_1(G) = \emptyset \iff \Delta^G \doteq \text{const.}$$

**3.3. The Alexander polynomial of a hyperplane arrangement.** Here is a quick application of this result. Let  $\mathcal{A}$  be a hyperplane arrangement in  $\mathbb{C}^\ell$ , with complement  $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ . If  $\mathcal{A}'$  is a generic two-dimensional section of  $\mathcal{A}$ , a well-known theorem guarantees that the inclusion  $X(\mathcal{A}') \hookrightarrow X(\mathcal{A})$  induces an isomorphism  $\pi_1(X(\mathcal{A}')) \xrightarrow{\cong} \pi_1(X(\mathcal{A}))$ . So, for the purpose of studying characteristic varieties and Alexander polynomials of arrangement groups, we may as well work with (affine) line arrangements.

**Example 3.3.** Let  $\mathcal{A}$  be a pencil of  $n$  lines through the origin of  $\mathbb{C}^2$ , defined by the equation  $x^n - y^n = 0$ . The fundamental group of the complement,  $G = \pi_1(X(\mathcal{A}))$ , is isomorphic to  $\mathbb{Z} \times F_{n-1}$ , where  $F_m$  denotes the free group on  $m$  generators. If  $n = 1$  or  $2$ , then  $\Delta^G = 1$  and  $\mathcal{V}_1(G) = \{1\}$ , but otherwise,  $\Delta^G = (t_1 \cdots t_n - 1)^{n-2}$  and  $\mathcal{V}_1(G) = \mathcal{W}_1(G) = \{t_1 \cdots t_n - 1 = 0\}$ .

This example is very special, though: in general, arrangement groups have trivial Alexander polynomial.

**Proposition 3.4.** *Let  $\mathcal{A}$  be an arrangement of  $n$  lines. The Alexander polynomial of  $G = \pi_1(X(\mathcal{A}))$  is non-constant if and only if  $\mathcal{A}$  is a pencil and  $n \geq 3$ .*

*Proof.* Suppose  $\mathcal{A}$  is not a pencil. By the above, each sub-arrangement which is a pencil of  $k \geq 3$  lines determines a ‘‘local’’ component of  $\mathcal{V}_1(G)$  of dimension  $k - 1$ . If  $n \leq 5$ , then it is readily seen that all components of  $\mathcal{V}_1(G)$  must be local. In general, there are other irreducible components, but each non-local component has dimension at most 4, by [19, Theorem 7.2]. Since  $\mathcal{A}$  is not a pencil, all components of  $\mathcal{V}_1(G)$  must have codimension at least 2, i.e.,  $\mathcal{W}_1(G) = \emptyset$ . Since  $b_1(G) \geq 2$ , Corollary 3.2(3) implies  $\Delta^G \doteq \text{const.}$   $\square$

**3.4. Parallel subtori.** Let  $H$  be a finitely generated abelian group. By a *subtorus* of  $\mathbb{T}_H$  we mean a subvariety  $W$  of the form

$$W = \rho \cdot S,$$

where  $\rho \in \mathbb{T}_H$  and  $S = f^*(\mathbb{T}_D)$ , for some epimorphism  $f: H \twoheadrightarrow D$  to a free abelian group  $D$ . Clearly, the translation factor  $\rho$  is not uniquely determined by  $W$ , while the subtorus  $S$  is. We call this subtorus the *direction* of  $W$ , and write  $S = \text{dir}(W)$ .

Two subtori,  $W, W' \subset \mathbb{T}_H$ , are said to be *parallel* if the direction of one is contained in the direction of the other, i.e.,

$$W \parallel W' \iff \text{dir}(W) \subset \text{dir}(W'), \text{ or } \text{dir}(W') \subset \text{dir}(W).$$

Clearly, two codimension-one subtori are parallel if and only if they have the same direction.

**3.5. Single essential variable.** Assume  $q := \text{rank } H$  is positive. In a suitable coordinate system  $(t_1, \dots, t_q)$  on  $\mathbb{T}_H^0 = (\mathbb{C}^*)^q$ , a codimension-one subtorus contained in  $\mathbb{T}_H^0$  is given by an equation of the form  $t_1 - z = 0$ , for some  $z \in \mathbb{C}^*$ . This observation motivates the following definition.

**Definition 3.5.** Let  $H$  be a finitely generated free abelian group. We say an element  $\Delta \in \mathbb{Z}H$  has a *single essential variable* if

$$\Delta \doteq \tilde{\nu}(P),$$

for some Laurent polynomial  $P \in \mathbb{Z}[\mathbb{Z}]$ , and some homomorphism  $\nu: \mathbb{Z} \rightarrow H$ .

Now assume  $q := \text{rank } H$  is positive. Note that, in the above definition,  $\nu$  may be supposed to be a split monomorphism. Identifying  $\mathbb{Z}H$  with the ring of Laurent polynomials,  $\Lambda_q = \mathbb{Z}[t_1^{\pm 1}, \dots, t_q^{\pm 1}]$ , Definition 3.5 can be understood in more concrete terms, as follows.

**Lemma 3.6.** *For a Laurent polynomial  $\Delta \in \Lambda_q$ , the following conditions are equivalent.*

- (1)  $\Delta$  has a single essential variable.
- (2) There is a Laurent polynomial  $P \in \mathbb{C}[\mathbb{Z}]$  and a group homomorphism  $\nu: \mathbb{Z} \rightarrow H$ , such that  $\Delta \doteq_{\mathbb{C}} \tilde{\nu}(P)$ .
- (3)  $\Delta(t_1, \dots, t_q) \doteq P(t_1^{e_1} \cdots t_q^{e_q})$ , for some polynomial  $P \in \mathbb{Z}[t^{\pm 1}]$ , and some coprime exponents  $e_1, \dots, e_q \in \mathbb{N}$ .
- (4) The Newton polytope of  $\Delta$  is a line segment.

The proof of this Lemma is left as an exercise in the definitions. We are now ready to connect this ‘1-dimensionality’ property of the Alexander polynomial to the geometry of the codimension-one stratum of the characteristic variety.

**Proposition 3.7.** *Let  $G$  be a finitely generated group, with  $b_1(G) \geq 2$ . The Alexander polynomial  $\Delta^G$  has a single essential variable if and only if either  $\mathcal{W}_1(G) = \emptyset$ , or all the irreducible components of  $\mathcal{W}_1(G)$  are parallel, codimension-one subtori of  $\mathbb{T}_G^0$ .*

*Proof.* By Corollary 3.2(3), we may assume that  $\Delta^G \neq \text{const}$  and  $\mathcal{W}_1(G) \neq \emptyset$ .

Set  $n = b_1(G)$ , and suppose  $\Delta^G \in \Lambda_n$  has a single essential variable. Then  $\Delta^G \doteq_{\mathbb{C}} \prod_j (t_1 - z_j)^{m_j}$ , for some pairwise distinct  $z_j \in \mathbb{C}^*$ . By Corollary 3.2(2),

$$(6) \quad \mathcal{W}_1(G) = \bigcup_j \{t_1 = z_j\},$$

as asserted.

Conversely, let  $\Delta^G \doteq_{\mathbb{C}} \prod_k f_k^{\mu_k}$  be the decomposition of  $\Delta^G$  into irreducible factors, leading to the decomposition into irreducible components for its zero locus,

$$(7) \quad V(\Delta^G) = \bigcup_k \{f_k = 0\}.$$

By assumption, (6) also holds, in a suitable coordinate system  $(t_1, \dots, t_n)$  on  $\mathbb{T}_G^0 = (\mathbb{C}^*)^n$ . Comparing decompositions (6) and (7), and making use of Corollary 3.2(2), we infer that  $\Delta^G \doteq_{\mathbb{C}} \prod_j (t_1 - z_j)^{\mu_j}$ . This finishes the proof.  $\square$

#### 4. QUASI-PROJECTIVE GROUPS

In this section, we analyze the characteristic variety  $\mathcal{V}_1(G)$  and the Alexander polynomial  $\Delta^G$  in the case when  $G$  can be realized as the fundamental group of a smooth, connected, quasi-projective complex variety.

**4.1. Admissible maps and isotropic subspaces.** We build on a foundational result of Arapura [2, Theorem V.1.6] on the structure of the first characteristic variety of a quasi-projective group, together with some refinements from [8] and [9].

**Theorem 4.1** (Arapura [2]). *Let  $G = \pi_1(M)$  be a quasi-projective group. Then all irreducible components  $W$  of  $\mathcal{V}_1(G)$  are subtori. More precisely, if  $\dim W > 0$ , then*

$$W = \rho \cdot f^*(\mathbb{T}_{\pi_1(C)}),$$

where  $\rho \in \mathbb{T}_G$  is a torsion element, and  $f: M \rightarrow C$  is a surjective regular map with connected generic fiber onto a smooth complex curve  $C$ .

A map  $f: M \rightarrow C$  as above is called *admissible*. Let  $F$  be the generic fiber of  $f$ , and  $i: F \rightarrow M$  the inclusion. The induced homomorphism,  $f_{\#}: \pi_1(M) \rightarrow \pi_1(C)$ , is surjective. Furthermore, the composite  $\pi_1(F) \xrightarrow{i_{\#}} \pi_1(M) \xrightarrow{\rho} \mathbb{C}^*$  is trivial. On the other hand, the sequence  $H_1(F) \xrightarrow{i_*} H_1(M) \xrightarrow{f_*} H_1(C) \rightarrow 0$  is not always exact in the middle. As shown in [8], the quotient

$$(8) \quad T(f) = \ker(f_*) / \text{im}(i_*)$$

is a finite abelian group. Furthermore, the components of  $\mathcal{V}_1(G)$  with direction  $f^*(\mathbb{T}_{\pi_1(C)})$  are parametrized by the Pontryagin dual  $\widehat{T}(f) = \text{Hom}(T(f), \mathbb{C}^*)$ , except when  $\chi(C) = 0$  (i.e.,  $C$  is either an elliptic curve, or  $C = \mathbb{C}^*$ ), in which case the trivial character must be excluded from  $\widehat{T}(f)$ .

We also need another definition, taken from [9]. Let

$$(9) \quad \cup_M: H^1(M; \mathbb{C}) \wedge H^1(M; \mathbb{C}) \rightarrow H^2(M; \mathbb{C}),$$

be the cup-product map. Given a subspace  $H \subset H^1(M; \mathbb{C})$ , denote by  $\cup_H$  the restriction of  $\cup_M$  to  $H \wedge H$ . We say that  $H$  is *0-isotropic* if  $\cup_H = 0$ , and *1-isotropic* if  $\dim \text{im}(\cup_H) = 1$  and  $\cup_H$  is a non-degenerate linear skew-form on  $H$ . Clearly, both notions depend only on  $G = \pi_1(M)$ . Note also that  $H^1(M; \mathbb{C})$  is naturally identified with the tangent space to  $\mathbb{T}_G$  at the identity,  $T_1\mathbb{T}_G$ .

**4.2. Position obstructions.** Our next result gives obstructions on the position of all components of  $\mathcal{V}_1(G)$ , thus extending Theorem B from [9], where only components through 1 were considered. Part (1) concerns the ambient position of each component in  $\mathbb{T}_G$ , while Parts (2)–(3) describe the relative position of each pair of components.

**Theorem 4.2.** *For an arbitrary quasi-projective group  $G$ , the following hold.*

- (1) *If  $W$  is an irreducible component of  $\mathcal{V}_1(G)$ , then  $W \subset \mathbb{T}_G$  is a subtorus, and the subspace  $T_1 \operatorname{dir}(W) \subset T_1 \mathbb{T}_G$  is either 0-isotropic or 1-isotropic.*
- (2) *If  $W$  and  $W'$  are two distinct components of  $\mathcal{V}_1(G)$ , then either  $\operatorname{dir}(W) = \operatorname{dir}(W')$ , or  $T_1 \operatorname{dir}(W) \cap T_1 \operatorname{dir}(W') = \{0\}$ .*
- (3) *For each pair of distinct components,  $W$  and  $W'$ , the intersection  $W \cap W'$  is a finite (possibly empty) set.*

*Proof.* Plainly, we may assume  $b_1(G) \geq 1$ , and consider only positive-dimensional components.

Part (1). We know from Theorem 4.1 that  $W$  is a subtorus of the form  $W = \rho \cdot S$ , with  $\rho$  a finite-order character, and  $S := \operatorname{dir}(W) = f^*(\mathbb{T}_{\pi_1(C)})$ , for some admissible map  $f: M \rightarrow C$ . If  $C$  is compact of positive genus and  $f^*: H^2(C; \mathbb{C}) \rightarrow H^2(M; \mathbb{C})$  is a monomorphism, then the subspace  $T_1 S = f^*(H^1(C; \mathbb{C}))$  is 1-isotropic; otherwise, this subspace is 0-isotropic.

Part (2). As above, write  $W' = \rho' \cdot S'$ , with  $S' = \operatorname{dir}(W') = f'^*(\mathbb{T}_{\pi_1(C')})$ , for some admissible map  $f': M \rightarrow C'$ . If  $W$  and  $W'$  are not parallel, then the arguments from [9, Lemmas 6.3–6.4] imply that  $S \cap S'$  must be finite, and so  $T_1 S \cap T_1 S' = \{0\}$ .

So suppose  $W$  and  $W'$  are parallel, but  $S \neq S'$ . We then have a strict inclusion, say  $S \subset S'$ . This forces  $\dim S' \geq 2$ , and so  $b_1(C') = \dim W' \geq 2$ . Hence, we can apply [8, Corollary 4.6] to the admissible map  $f': M \rightarrow C'$  and the rank one local system  $\mathcal{L}_1 = \mathbb{C}_\rho$ .<sup>1</sup> Set  $W_1 = \rho \cdot S'$ . Since  $W \subseteq W_1$ , the restriction of  $\mathcal{L}_1$  to the generic fiber of  $f'$  must be trivial. Thus,  $\mathcal{L}_1$  corresponds to a character  $\alpha \in \widehat{T}(f')$ . There are two cases to consider.

*Case 1.* *Either  $C'$  is not an elliptic curve, or  $C'$  is an elliptic curve and the character  $\alpha$  is non-trivial.* Then, according to [8, Corollary 5.8],  $W_1$  is an irreducible component of  $\mathcal{V}_1(G)$ . Since  $W$  is also an irreducible component of  $\mathcal{V}_1(G)$ , it follows that  $W = W_1$ , and hence  $S = S'$ , a contradiction.

*Case 2.*  *$C'$  is an elliptic curve and the character  $\alpha$  is trivial.* Then, again by [8, Corollary 5.8],  $W_1 = S'$  is not an irreducible component of  $\mathcal{V}_1(G)$ . In this case  $\dim S' = 2$  and the elements in  $T_1 S' = f'^*(H^1(C'; \mathbb{C}))$  have Hodge type (1,0) and (0,1). On the other hand,  $\dim S = 1$ , which implies that  $C = \mathbb{C}^*$ , and the elements in  $T_1 S = f^*(H^1(\mathbb{C}^*; \mathbb{C}))$  have Hodge type (1,1). This contradicts  $T_1 S \subset T_1 S'$ .

Part (3). This is an immediate consequence of Part (2). □

Note that, in Part (1), both 0-isotropic and 1-isotropic subspaces may arise; see [9]. The second situation described in Part (2) often occurs for complements of complex

<sup>1</sup>In [8], “for a generic local system” actually means “for all but finitely many local systems”.

hyperplane arrangements, see for instance [22]. The first situation often occurs for complements of Seifert links, see for instance Example 7.3.

**4.3. Alexander polynomial and quasi-projectivity.** The relative position obstruction from Theorem 4.2(2) translates into restrictions on the multivariable Alexander polynomial of a quasi-projective group, as follows.

**Theorem 4.3.** *Let  $G = \pi_1(M)$  be the fundamental group of a smooth, connected, complex quasi-projective variety.*

- (1) *If  $b_1(G) \neq 2$ , then the Alexander polynomial  $\Delta^G$  has a single essential variable.*
- (2) *If  $b_1(G) \geq 2$ , and  $\Delta^G$  has a single essential variable, then either*
  - (a)  *$\Delta^G = 0$ , or*
  - (b)  *$\Delta^G(t_1, \dots, t_n) \doteq_{\mathbb{C}} P(u)$ , where  $P$  is a product of cyclotomic polynomials (possibly equal to 1), and  $u = t_1^{e_1} \cdots t_n^{e_n}$ , with  $\gcd(e_1, \dots, e_n) = 1$ .*
- (3) *If  $M$  is actually a projective variety, then  $\Delta^G \doteq \text{const}$ .*

*Proof.* In Part (1), we may assume  $b_1(G) \geq 3$ . The claim then follows from Proposition 3.7 and Theorem 4.2(2).

In Part (2), we may assume—after a change of variables if necessary—that  $\Delta^G \doteq P(t_1)$ , with  $P$  a non-constant polynomial in  $\mathbb{Z}[t_1]$ . By Corollary 3.2(2) and Arapura’s Theorem 4.1, all roots of  $P$  are roots of unity.

In Part (3), since the odd Betti numbers of  $M$  are even, we may assume  $b_1(G) \geq 2$ , for, otherwise, clearly  $\Delta^G$  is constant. By Corollary 3.2(3), we are left with showing that  $\mathcal{W}_1(G) = \emptyset$ . Suppose, to the contrary, that  $\mathcal{V}_1(G)$  has a codimension-one component, call it  $W$ ; then there is an admissible map,  $f: M \rightarrow C$ , onto a projective curve, such that  $\text{dir}(W) = f^*(\mathbb{T}_{\pi_1(C)})$ . This implies  $b_1(C)$  is odd, a contradiction.  $\square$

**Corollary 4.4.** *If  $G$  is a quasi-projective group with  $n = b_1(G) \geq 3$ , then*

$$\Delta^G(t_1, \dots, t_n) \doteq cP(t_1^{e_1} \cdots t_n^{e_n}),$$

where  $P \in \mathbb{Z}[t]$  is a product of cyclotomic polynomials,  $c \in \mathbb{Z}$ , and  $\gcd(e_1, \dots, e_n) = 1$ .

**Remark 4.5.** A result similar to Theorem 4.3(2) was obtained by Libgober in [15] for complements of plane algebraic curves,  $M = \mathbb{P}^2 \setminus C$ . Yet his result only holds for the single variable Alexander polynomial,  $\Delta = \Delta_{\phi_{\text{lk}}}^G$ , associated to  $G = \pi_1(M)$  and the total linking homomorphism,  $\phi_{\text{lk}}: G \rightarrow \mathbb{Z}$ . Note that the polynomial  $\Delta(t) \in \mathbb{Z}[t^{\pm 1}]$  carries information only on the intersection of a one-dimensional subtorus of  $\mathbb{T}_G$  with  $\mathcal{V}_1(G)$ , whereas our general result from Theorem 4.3, when combined with Proposition 3.7, puts severe restrictions on the global structure of  $\mathcal{V}_1(G)$ .

**4.4. The case  $b_1 = 2$ .** Concerning Part (1) of Theorem 4.3, we now give an example showing that the Alexander polynomials of quasi-projective groups with  $b_1 = 2$  do exhibit an exceptional behavior.

**Example 4.6.** Consider the surface  $X: xy - z^2 = 0$  in  $\mathbb{C}^3$ . Blowing up the origin, we obtain a smooth surface  $Y$ , which is known to be the total space of the line bundle

$\mathcal{O}(-2)$  on  $\mathbb{P}^1$ . Hence  $\tilde{H}_0(Y; \mathbb{Z}) = H_1(Y; \mathbb{Z}) = 0$ . Indeed, the smooth projective conic  $X': xy - z^2 = 0$  in  $\mathbb{P}^2$  is rational of degree 2 (use the genus-degree formula). Consider the lines  $L_1: x = 1$  and  $L_2: y = 1$  on  $X$  and denote by  $C_1$  and  $C_2$  their proper transforms in  $Y$ . Since the lines  $L_j$  do not pass through the origin, the curves  $C_j$  are isomorphic to  $L_j$ , therefore to  $\mathbb{C}$ .

Let  $M = Y \setminus (C_1 \cup C_2)$ . The above discussion implies that  $H_1(M; \mathbb{Z}) = \mathbb{Z}^2$ , with basis given by some small loops  $\gamma_1$  and  $\gamma_2$  about  $C_1$  and  $C_2$ . Consider the group  $G = \pi_1(M)$ . Clearly,  $G$  is quasi-projective, and  $b_1(G) = 2$ . We claim that the Alexander polynomial of  $G$  does *not* have a single essential variable.

Indeed, let  $C = \mathbb{C} \setminus \{1\}$ , and consider the admissible map  $f: M \rightarrow C$ , obtained by composing the projection map  $X \setminus (L_1 \cup L_2) \rightarrow C$ ,  $(x, y, z) \mapsto x$  with the blow-up morphism. The induced homomorphism,  $f^*: H^1(C; \mathbb{Z}) \rightarrow H^1(M; \mathbb{Z})$ , has image equal to  $\mathbb{Z} \cdot \gamma_1^*$ . Recall from §4.1 that  $T(f)$  denotes the finite abelian group  $\ker(f_*) / \text{im}(i_*)$ , where  $i: F \hookrightarrow M$  is the inclusion of the generic fiber of  $f$ . In our situation, a computation shows that  $T(f) = \mathbb{Z}/2\mathbb{Z}$ . It follows from [8] that there is a unique 1-dimensional component of  $\mathcal{V}_1(G)$  associated to  $f$ , call it  $W_f$ , with tangent direction  $\gamma_1^*$ . Proceeding in exactly the same way, starting from the other projection map,  $g: (x, y, z) \mapsto y$ , we obtain another 1-dimensional component, call it  $W_g$ , with tangent direction  $\gamma_2^*$ . Clearly,  $W_f \nparallel W_g$ . By Proposition 3.7,  $\Delta^G$  does not have a single essential variable.

**4.5. Boundary manifolds of line arrangements.** As an application of our methods, we now give a class of examples where the restrictions imposed by Theorem 4.3 completely settle Serre's problem. More applications will be given in a forthcoming paper.

Let  $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$  be an arrangement of lines in  $\mathbb{P}^2$ . Of course, the complement  $X(\mathcal{A}) = \mathbb{P}^2 \setminus \bigcup_{i=0}^n \ell_i$  is a smooth, quasi-projective variety, and so  $\pi_1(X(\mathcal{A}))$  is a quasi-projective group. On the other hand, consider the closed, orientable 3-manifold  $M(\mathcal{A})$ , defined as the boundary of a regular neighborhood of the curve  $C = \bigcup_{i=0}^n \ell_i$  in  $\mathbb{P}^2$ . The fundamental group of the boundary manifold,  $G_{\mathcal{A}} = \pi_1(M_{\mathcal{A}})$ , admits a commutator-relators presentation, with  $b_1(G_{\mathcal{A}}) \geq n$ , see [4].

**Proposition 4.7.** *The group  $G_{\mathcal{A}}$  is quasi-projective if and only if  $\mathcal{A}$  is a pencil or a near-pencil.*

*Proof.* If  $\mathcal{A}$  is a pencil of  $n + 1$  lines, then  $G_{\mathcal{A}}$  is the free group of rank  $n$ ; if  $\mathcal{A}$  is a near-pencil (i.e., a pencil of  $n$  lines, together with an extra line in general position), then  $G_{\mathcal{A}}$  is the product of  $\mathbb{Z}$  with the fundamental group of a compact Riemann surface of genus  $n - 1$ , see [4]. In either case,  $G_{\mathcal{A}}$  is quasi-projective.

For the converse, let  $\mathcal{A}$  be an arrangement that is not a pencil or a near-pencil; in particular,  $n \geq 3$ . By [4, Theorem 5.2], the Alexander polynomial  $\Delta^{G_{\mathcal{A}}}$  is a product of (irreducible) factors of the form  $t_{i_1} \cdots t_{i_r} - 1$ ; by [4, Proposition 5.5],  $\Delta^{G_{\mathcal{A}}}$  must have at least two distinct such factors.

Now suppose  $G_{\mathcal{A}}$  is quasi-projective. Since  $b_1(G_{\mathcal{A}}) \geq 3$ , we infer from Theorem 4.3(1) that  $\Delta^{G_{\mathcal{A}}}$  must have a single essential variable. By Proposition 3.7 and Corollary 3.2(2),

the components of  $V(\Delta^{G_A})$  must be parallel, codimension-one subtori. Since there are at least two such subtori containing 1, we reach a contradiction.  $\square$

## 5. MULTIPLICITIES, TWISTED BETTI RANKS, AND GENERIC BOUNDS

We saw in Section 3 that the *reduced* Alexander polynomial of  $G$  determines the codimension-one stratum or the first characteristic variety,  $\mathcal{W}_1(G)$ . In this section, we pursue this approach, analyzing the connection between the multiplicities of the factors of  $\Delta^G$  and the higher-depth characteristic varieties,  $\mathcal{V}_k(G)$ .

**5.1. Generic Betti ranks.** Let  $G$  be a finitely generated group. Set  $\Lambda := \mathbb{C}[G_{\text{abf}}]$ . Consider the irreducible subvariety  $V(\mathfrak{p}) \subset \mathbb{T}_G^0$  associated to a prime ideal  $\mathfrak{p} \subset \Lambda$ .

**Definition 5.1.** The *generic Betti number* of  $G$  relative to  $\mathfrak{p}$  is

$$b_1^{\text{gen}}(G, \mathfrak{p}) = \max\{k \mid V(\mathfrak{p}) \subset \mathcal{V}_k(G)\}.$$

When  $\mathfrak{p} = (f)$ , we abbreviate  $b_1^{\text{gen}}(G, (f))$  to  $b_1^{\text{gen}}(G, f)$ .

For a character  $\rho \in \mathbb{T}_G$ , set  $b_1(G, \rho) = \dim_{\mathbb{C}} H_1(G; \mathbb{C}_\rho)$ . We then have

$$(10) \quad b_1(G, \rho) = b_1^{\text{gen}}(G, \mathfrak{p}),$$

for  $\rho$  in the Zariski open, non-empty subset  $V(\mathfrak{p}) \setminus \mathcal{V}_{b+1}(G)$ , where  $b = b_1^{\text{gen}}(G, \mathfrak{p})$ . See also Farber [11, Theorem 1.5] for another interpretation. As the next example shows, generic Betti numbers are *not* always determined by the Alexander polynomial.

**Example 5.2.** By a classical result of Lyndon (see [12]), one may construct groups with quite complicated Alexander matrices. Denote by  $F_m$  the free group on  $m$  generators. Suppose  $v_1, \dots, v_m$  are elements in  $\mathbb{Z}[\mathbb{Z}^m]$  satisfying  $\sum_{j=1}^m (x_j - 1)v_j = 0$ . There exists then an element  $r \in F'_m$  such that  $v_j = (\partial r / \partial x_j)^{\text{ab}}$ , for all  $j$ .

Given any two elements  $\Phi, \Psi \in \mathbb{Z}[\mathbb{Z}^3]$ , we may construct in this way a commutator-relators group,  $G = \langle x_1, x_2, x_3 \mid r_1, r_2 \rangle$ , with Alexander matrix

$$A_G \begin{pmatrix} (x_2 - 1)\Phi & (1 - x_1)\Phi & 0 \\ 0 & (x_3 - 1)\Psi & (1 - x_2)\Psi \end{pmatrix}.$$

An easy computation reveals that  $\Delta^G = (x_2 - 1)\Phi\Psi$ .

Now let  $G_1$  and  $G_2$  be groups corresponding to  $\Phi_1 = \Psi_1 = \alpha\beta$ , respectively  $\Phi_2 = \alpha^2$ ,  $\Psi_2 = \beta^2$ , where  $\alpha = x_1x_2 + 1$  and  $\beta = x_2x_3 + 1$ . These two groups share the same Alexander polynomial,  $\Delta = (x_2 - 1)\alpha^2\beta^2$ . On the other hand,  $b_1^{\text{gen}}(G_1, \alpha) \geq 2$ , whereas  $b_1^{\text{gen}}(G_2, \alpha) \leq 1$ .

**5.2. Generic upper bounds.** Nevertheless, as the next result shows, the multiplicities of the Alexander polynomial provide useful upper bounds for the generic Betti numbers.

First, some notation, which will be needed in the proof. Let  $\mathcal{O}_n$  be the ring of germs of analytic functions at  $0 \in \mathbb{C}^n$ . The *order* of a germ  $f \in \mathcal{O}_n$  at 0, denoted  $\nu_0(f)$ , is the degree of the first non-vanishing homogeneous term in the Taylor expansion of  $f$  at 0.

**Theorem 5.3.** *Let  $\Delta^G \doteq_{\mathbb{C}} f_1^{\mu_1} \cdots f_s^{\mu_s}$  be the decomposition into irreducible factors of the Alexander polynomial of a finitely generated group  $G$ . Then, for each  $j$ ,*

$$b_1^{\text{gen}}(G, f_j) \leq \mu_j.$$

*Proof.* By the discussion from §2.6, we may assume  $G$  admits a finite presentation, say  $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_h \rangle$ . Set  $n = b_1(G)$ ,  $\Delta = \Delta^G$ , and identify  $\mathbb{Z}G_{\text{abf}} = \Lambda_n$ .

Fix an index  $j \in \{1, \dots, s\}$ , and put  $W = V(f_j)$ . To prove the desired inequality, it is enough to find a Zariski open, non-empty subset  $U \subset W$  such that  $b_1(G, \rho) \leq \mu_j$ , for every  $\rho \in U$ .

By definition of  $\Delta^G$ , there exist  $g_1, \dots, g_p \in \Lambda_n$  with  $\gcd(g_1, \dots, g_p) = 1$  such that  $\mathcal{E}_1(A_G) = (\Delta^G \cdot g_1, \dots, \Delta^G \cdot g_p)$ . Define

$$(11) \quad U = W \setminus (\{1\} \cup \Sigma \cup Z \cup Y),$$

where  $Y = V(g_1) \cap \cdots \cap V(g_p)$ , the set  $\Sigma$  is the singular part of  $W$ , and  $Z$  is the union of those irreducible components of  $\mathcal{V}_1(G)$  contained in  $\mathbb{T}_G^0$ , yet different from  $W$ .

Clearly,  $U$  is a Zariski open set. To show that  $U$  is non-empty, we need to verify  $W \not\subset \{1\} \cup \Sigma \cup Z \cup Y$ . If  $W \subset \{1\}$  then necessarily  $n = 1$  and  $\Delta(1) = 0$ , which implies  $n \geq 2$ , a contradiction. Obviously,  $W \not\subset \Sigma$ . By Corollary 3.2(2), each  $V(f_i)$  is a component of  $\mathcal{V}_1(G)$  contained in  $\mathbb{T}_G^0$ ; in particular,  $W \not\subset Z$ . Finally,  $W \subset Y$  would imply that  $f_j$  divides each  $g_i$ , which is impossible.

Now fix  $\rho \in U$ . Since  $\rho \in W \setminus (\Sigma \cup Z)$ , we may find analytic coordinates in a neighborhood  $U_0 \subset U$  of  $\rho$ , say  $(z, w)$  with  $z \in \mathcal{O}_n$  and  $w \in \mathcal{O}_n^{n-1}$ , such that, in  $U_0$ ,

$$W = \{z = 0\}, \quad \{w = 0\} \cap \mathcal{V}_1(G) \setminus \{\rho\}, \quad \left. \frac{d}{dz} \right|_{z=0} f_j(z, 0) \neq 0.$$

In particular,  $\nu_0 f_j(z, 0) = 1$ .

Let  $A_G: \Lambda_n^h \rightarrow \Lambda_n^m$  be the Alexander matrix of  $G$ . The base change  $\tau: \Lambda_n \rightarrow \mathcal{O}_1$ , given by restriction of Laurent polynomials to  $\{w = 0\}$ , yields the ‘local’ Alexander matrix  $A_G^\tau: \mathcal{O}_1^h \rightarrow \mathcal{O}_1^m$ . For  $t \in (\mathbb{C}, 0)$  belonging to the local transversal slice to  $W$  at  $\rho$  given by  $\{w = 0\}$ , since  $\rho \neq 1$ , we have

$$(12) \quad 1 + b_1(G, t) = \dim_{\mathbb{C}} \text{coker } A_G^\tau(t),$$

where  $A_G^\tau(t) = \mathbb{C}_t \otimes_{\mathcal{O}_1} A_G^\tau$  denotes the evaluation of the Alexander matrix at  $t$ . On the other hand, since  $\mathcal{O}_1$  is a PID,

$$(13) \quad \text{coker } A_G^\tau = \mathcal{O}_1^r \oplus \left( \bigoplus_{k=1}^{\ell} \mathcal{O}_1 / z^{\nu_k} \mathcal{O}_1 \right),$$

for some integers  $r \geq 0$  and  $\nu_k > 0$ . By comparing (12) and (13), we infer that  $r = 1$  (by taking  $t \neq 0$ ), and  $b_1(G, 0) = \ell \geq 1$  (by taking  $t = 0$ ).

Now, since  $\rho \notin Y$ , we must have  $g_i(\rho) \neq 0$ , for some  $i$ . Thus  $\tau(\Delta) \in \mathcal{E}_1(\text{coker } A_G^\tau)$ , by base change. We deduce from formula (2) that  $\tau(\Delta) \in z^{\nu_1 + \cdots + \nu_\ell} \mathcal{O}_1$ . In particular,  $\nu_0 \Delta(z, 0)$  is at least  $b_1(G, 0) = b_1(G, \rho)$ .

Making use again of (11), we see that  $f_i(\rho) \neq 0$ , for  $i \neq j$ ; hence,  $\nu_0\Delta(z, 0) = \mu_j \cdot \nu_0 f_j(z, 0)$ . But  $\nu_0 f_j(z, 0) = 1$ , and so  $\mu_j \geq b_1(G, \rho)$ . This finishes the proof.  $\square$

**5.3. Discussion of the generic bound.** The inequality from Theorem 5.3 may well be strict, as the following example shows.

**Example 5.4.** Let  $f \in \mathbb{Z}[\mathbb{Z}^2]$  be a Laurent polynomial, irreducible over  $\mathbb{C}$ . Using the method described in Example 5.2, we can construct, for each  $k \geq 1$ , a group  $G_k = \langle x_1, x_2 \mid r_k \rangle$  with Alexander polynomial  $\Delta^{G_k} = f^k$ . Clearly,  $b_1^{\text{gen}}(G_k, f) = 1$ , while the multiplicity of  $f$  as a factor of  $\Delta^{G_k}$  is  $\mu = k$ .

The next example shows that the inequality  $b_1^{\text{gen}}(G, f_j) \leq \mu_j$  is sharp.

**Example 5.5.** Let  $G = \mathbb{Z} \times F_{n-1}$  be the fundamental group of the complement of a pencil of  $n \geq 3$  lines in  $\mathbb{C}^2$ . Recall from Example 3.3 that  $\Delta^G = f^{n-2}$ , where  $f = t_1 \cdots t_n - 1$ . It is readily checked that  $b_1^{\text{gen}}(G, f) = n - 2$ , which matches the multiplicity of  $f$  as a factor of  $\Delta^G$ .

Finally, let us note that the inequality from Theorem 5.3 may well fail for *non-generic* twisted Betti ranks.

**Example 5.6.** Let  $G = \langle x_1, x_2, x_3 \mid r_1, r_2 \rangle$ , be the group  $G_1$  from Example 5.2, with Alexander polynomial  $\Delta^G = (x_2 - 1)\Phi\Psi$ . Consider the character  $\rho = (-1, 1, -1)$ , belonging to  $V(x_2 - 1) \subset \mathcal{W}_1(G)$ . A computation shows that  $b_1(G, \rho) = 2$ , whereas the multiplicity of the factor  $x_2 - 1$  in  $\Delta^G$  is 1.

## 6. ALMOST PRINCIPAL ALEXANDER IDEALS AND MULTIPLICITY BOUNDS

In this section, we delineate a class of groups  $G$  (which includes the above example), for which the Alexander polynomial may be used to produce another upper bound for  $b_1(G, \rho)$ , valid for *all* nontrivial local systems in  $\mathbb{T}_G^0$ .

**6.1. Almost principal ideals.** We start with a definition, inspired by work of Eisenbud-Neumann [10] and McMullen [17].

**Definition 6.1.** Let  $G$  be a finitely generated group. We say that the Alexander ideal  $\mathcal{E}_1(A_G)$  is *almost principal* if there exists an integer  $d \geq 0$  such that

$$I^d \cdot (\Delta^G) \subset \mathcal{E}_1(A_G),$$

over  $\mathbb{C}$ , where  $I$  denotes the augmentation ideal of the group ring  $\mathbb{Z}[G_{\text{abf}}]$ .

For this class of groups, the reduced Alexander polynomial determines  $\mathcal{V}_1(G) \cap \mathbb{T}_G^0$ , since  $\mathcal{V}_1(G) \cap \mathbb{T}_G^0 = V(\Delta^G)$ , away from 1, as noted in Proposition 2.4. There is an abundance of interesting examples of this kind.

Recall that the *deficiency* of a finitely presented group  $G$ , written  $\text{def}(G)$ , is the supremum of the difference between the number of generators and the number of relators, taken over all finite presentations of  $G$ .

**Lemma 6.2.** *If  $b_1(G) = 1$ , or  $\text{def}(G) > 0$ , then  $\mathcal{E}_1(A_G)$  is almost principal.*

*Proof.* If  $b_1(G) = 1$ , then  $\Lambda_1(\mathbb{C})$  is principal, and so  $\mathcal{E}_1(A_G) = (\Delta^G)$ . If  $b_1(G) \geq 2$  and  $\text{def}(G) > 0$ , then  $\mathcal{E}_1(A_G) = I \cdot (\Delta^G)$ , by Theorems 6.1 and 6.3 in [10].  $\square$

Another class of groups with this property occurs in low-dimensional topology.

**Lemma 6.3.** *Let  $M$  be a compact, connected 3-manifold, and let  $G = \pi_1(M)$ . If either  $\partial M \neq \emptyset$  and  $\chi(\partial M) = 0$ , or  $\partial M = \emptyset$  and  $M$  is orientable, then  $\mathcal{E}_1(A_G)$  is almost principal.*

*Proof.* In the first case,  $\text{def}(G) > 0$ , by [10, Lemma 6.2], and so the result follows from Lemma 6.2. In the second case,  $I^2 \cdot (\Delta^G) \subset \mathcal{E}_1(A_G)$ , as shown in [17, Theorem 5.1].  $\square$

**6.2. Multiplicity bounds for  $b_1(G, \mathbb{C}_\rho)$ .** We are now ready to state our result concerning multiplicity bounds for the twisted Betti numbers for the class of groups delineated above.

For a character  $\rho \in (\mathbb{C}^*)^n$ , and a Laurent polynomial  $f \in \Lambda_n$ , denote by  $\nu_\rho(f)$  the order of vanishing of the germ of  $f$  at  $\rho$ .

**Theorem 6.4.** *Let  $G$  be a finitely generated group, and let  $\Delta^G \doteq_{\mathbb{C}} f_1^{\mu_1} \cdots f_s^{\mu_s}$  be the decomposition into irreducible factors of its Alexander polynomial. Assume the Alexander ideal of  $G$  is almost principal. If  $\rho \in \mathbb{T}_G^0 \setminus \{1\}$ , then*

$$(14) \quad \dim_{\mathbb{C}} H_1(G; \mathbb{C}_\rho) \leq \sum_{j=1}^s \mu_j \cdot \nu_\rho(f_j).$$

*Proof.* As before, we may assume  $G$  admits a finite presentation, say  $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_h \rangle$ . Set  $n = b_1(G)$ , and consider the change of rings  $\psi: \Lambda_n \rightarrow \mathcal{O}_n$  given by restriction of Laurent polynomials near a fixed  $\rho \in \mathbb{T}_G^0 \setminus \{1\}$ . Let  $\mathbf{A}_G^\psi: \mathcal{O}_n^h \rightarrow \mathcal{O}_n^m$  be the corresponding ‘local’ Alexander matrix, and  $\mathbf{A}_G(\rho): \mathbb{C}^h \rightarrow \mathbb{C}^m$  its evaluation at  $\rho$ . Set  $r = \text{rank } \mathbf{A}_G(\rho)$  and  $b = b_1(G, \rho)$ . We then have

$$(15) \quad \dim_{\mathbb{C}} \text{coker } \mathbf{A}_G(\rho) = m - r = b + 1,$$

since  $\rho \neq 1$ .

Setting  $Z = \ker \mathbf{A}_G(\rho)$ , we may choose vector space decompositions,  $\mathbb{C}^h = Z \oplus \mathbb{C}^r$  and  $\mathbb{C}^m = N \oplus \mathbb{C}^r$ , such that  $\mathbf{A}_G(\rho)$  takes the diagonal form  $0 \oplus \text{id}$ . In the corresponding decomposition over  $\mathcal{O}_n$ , the  $r \times r$  submatrix of  $\mathbf{A}_G^\psi$  is invertible. Using (15), we see that

$$(16) \quad \text{coker } \mathbf{A}_G^\psi = \text{coker } (D: \mathcal{O}_n^{h-r} \rightarrow \mathcal{O}_n^{b+1}),$$

where  $D$  is equivalent to the zero matrix, modulo the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}_n$ .

By assumption, the Alexander ideal of  $G$  is almost principal, i.e.,  $I^d \cdot (\Delta^G) \subset \mathcal{E}_1(A_G)$ , for some  $d$ . Since  $\rho \neq 1$ , we obtain by base change

$$\psi(\Delta^G) \in \mathcal{E}_1(\text{coker } \mathbf{A}_G^\psi).$$

Using (16), we deduce that  $\psi(\Delta^G) \in \mathfrak{m}^b$ . Hence,  $b \leq \nu_\rho(\Delta^G) = \sum_j \mu_j \cdot \nu_\rho(f_j)$ , as asserted.  $\square$

**Corollary 6.5.** *Assume in Theorem 6.4 that  $G_{\text{ab}}$  is torsion-free. If the upper bound (14) is attained for every  $\rho \in \mathbb{T}_G^0 \setminus \{1\}$ , then the Alexander polynomial  $\Delta^G$  determines the characteristic varieties  $\mathcal{V}_k(G)$ , for all  $k \geq 1$ .*

**6.3. Discussion.** As the next example shows, the hypothesis that the character  $\rho$  be non-trivial is essential for Theorem 6.4 to hold.

**Example 6.6.** Let  $G = \mathbb{Z} \times F_{n-1}$ , where  $n \geq 3$ . Note that  $\text{def}(G) = 1$ , and so, by Lemma 6.2, the Alexander ideal of  $G$  is almost principal. Clearly,  $b_1(G, 1) = n$ , while the upper bound from Theorem 6.4 equals  $(n-2) \cdot \nu_1(t_1 \cdots t_n - 1) = n-2$ .

The hypothesis on the Alexander ideal being almost principal is also needed.

**Example 6.7.** The method discussed in Example 5.2 produces a commutator-relators group,  $G = \langle x_1, x_2, x_3 \mid r_1, r_2, r_3 \rangle$ , with Alexander matrix

$$A_G \begin{pmatrix} (x_2 - 1)\Phi & (1 - x_1)\Phi & 0 \\ 0 & (x_3 - 1)\Phi & (1 - x_2)\Phi \\ (x_3 - 1)\Psi & 0 & (1 - x_1)\Psi \end{pmatrix},$$

where  $\Phi = x_1x_3 - 1$  and  $\Psi = x_1x_2 - 1$ . As is readily seen,  $\Delta^G \doteq \Phi$ . For the nontrivial local system  $\rho = (1, -1, 1)$ , we have  $b_1(G, \rho) = 2$ ; on the other hand, the corresponding upper bound from (14) is 1.

Inequality (14) may well be strict, as the next example shows.

**Example 6.8.** The groups  $G_k$ , with  $k > 1$ , from Example 5.4 meet the requirements from Theorem 6.4. If  $\rho \in V(f) \setminus \{1\}$ , then clearly  $b_1(G_k, \rho) = 1 < k\nu_\rho(f)$ .

Finally, let us point out that inequality (14) is sharp.

**Example 6.9.** Let  $G = \mathbb{Z} \times F_{n-1}$  be the group from Example 3.3, with Alexander polynomial  $\Delta^G = f^{n-2}$ , where  $f = t_1 \cdots t_n - 1$ . Clearly,  $G$  satisfies the assumptions of Theorem 6.4. If  $\rho \in V(f) \setminus \{1\}$ , then  $b_1(G, \rho) = (n-2) \cdot \nu_\rho(f) = n-2$ , i.e., the upper bound (14) is attained.

**6.4. The bounds in a special case.** When  $\Delta^G \doteq_{\mathbb{C}} \prod_j f_j^{\mu_j}$  has a single essential variable, Theorem 6.4 takes the following simpler form.

**Corollary 6.10.** *Suppose the Alexander ideal of  $G$  is almost principal,  $\Delta^G$  has a single essential variable, and  $\Delta^G \not\equiv \text{const}$ . Then*

- (1) *The intersection  $\mathcal{V}_1(G) \cap \mathbb{T}_G^0 \setminus \{1\}$  equals the disjoint union  $\coprod_j V(f_j) \setminus \{1\}$ .*
- (2) *If  $1 \neq \rho \in V(f_j)$ , then  $1 \leq b_1(G, \rho) \leq \mu_j$ .*

*Proof.* Part (1) follows by putting together Definition 3.5 and the remark made after Definition 6.1. As for Part (2), use the obvious fact that  $\nu_\rho(t_1 - z) = 1$ , for any  $\rho \in V(t_1 - z)$ .  $\square$

**Remark 6.11.** Let  $G$  be a group as in Corollary 6.10. Pick any nontrivial element  $\rho \in V(f_j)$ , and let  $N$  be a local transversal slice to  $V(f_j)$  at  $\rho$ . In this situation, the arguments from the proof of Theorem 5.3 work not only for generic elements of  $V(f_j)$ , but also for arbitrary  $\rho$  in  $\mathbb{T}_G^0 \setminus \{1\}$ , giving the following information.

In (13),  $r = 1$  and  $b_1(G, 0) = \ell$ . Due to the fact that  $\mathcal{E}_1(A_G)$  is almost principal, and  $\rho \neq 1$ , we have  $(\tau(\Delta^G)) = \mathcal{E}_1(\text{coker } \mathbf{A}_G^\tau) = (z^{\nu_1 + \dots + \nu_\ell})$ . This implies

$$\mu_j = \nu_0 \Delta^G(z, 0) = \sum_{k=1}^{\ell} \nu_k \geq \ell = b_1(G, \rho).$$

Therefore, the equality  $b_1(G, \rho) = \mu_j$  is equivalent to

$$(17) \quad \nu_1 = \dots = \nu_\ell = 1,$$

or, in invariant form,  $z \cdot (\mathcal{O}_1(\rho, N) \otimes_{\mathbb{C}[G_{\text{abf}}]} B_G \otimes \mathbb{C}) = 0$ , where  $\mathcal{O}_1(\rho, N)$  denotes  $\mathcal{O}_1$ , with module structure given by restriction of Laurent polynomials to  $N$ .

We shall see in Theorem 7.2 that conditions (17) are satisfied by Seifert links. On the other hand, these conditions are not satisfied in general, as the next example shows (see also Examples 6.14 and 6.16 below).

**Example 6.12.** Let  $G = \langle x_1, x_2 \mid r_1, r_2 \rangle$  be a commutator-relators group, with  $b_1(G) = 2$  and Alexander matrix

$$\mathbf{A}_G = \begin{pmatrix} (x_1 - 1)(x_2 - 1)\alpha & -(x_1 - 1)^2\alpha \\ (x_2 - 1)^2\alpha & -(x_1 - 1)(x_2 - 1)\alpha \end{pmatrix},$$

where  $\alpha = (x_1 x_2 + 1)^k$ , and  $k > 1$ . In this case,  $\Delta^G(x_1, x_2) \doteq \alpha$ ; in particular,  $\Delta^G$  has a single essential variable. Moreover,  $I^2 \cdot (\Delta^G) \subset \mathcal{E}_1(A_G)$ , and so the Alexander ideal is almost principal. Now, for any  $\rho \in V(x_1 x_2 + 1) \subset \mathbb{T}_G \setminus \{1\}$ , we see that  $b_1(G, \rho) = 1$ , even though the multiplicity of the factor  $x_1 x_2 + 1$  of  $\Delta^G$  is  $k > 1$ .

**6.5. Monodromy.** For the rest of this section, we assume  $b_1(G) = 1$ . In this case, the torsion-free abelianization map is  $\phi_{\text{abf}}: G \rightarrow \mathbb{Z}$ . Identify the group ring  $\mathbb{Z}\mathbb{Z}$  with  $\Lambda_1 = \mathbb{Z}[t^{\pm 1}]$ , and note that  $I_{\mathbb{Z}} = (t - 1)\Lambda_1$  is a free  $\Lambda_1$ -module of rank 1. The Crowell exact sequence (4) then yields a decomposition  $A_G \cong B_G \oplus I_{\mathbb{Z}}$ , from which we infer

$$(18) \quad \mathcal{E}_1(A_G) = \mathcal{E}_0(B_G).$$

Obviously, the Alexander polynomial  $\Delta^G$  has a single variable. Since  $\Lambda := \Lambda_1 \otimes \mathbb{C}\mathbb{C}[t^{\pm 1}]$  is a PID, the ideal  $\mathcal{E}_1(A_G) \otimes \mathbb{C}$  is principal (generated by  $\Delta^G$ ). Hence, Corollary 6.10 applies, provided  $\Delta^G \neq \text{const}$ . In view of (18), this condition is equivalent to  $B_G \otimes \mathbb{C}$  being a non-zero, torsion  $\Lambda$ -module, in which case

$$(19) \quad B_G \otimes \mathbb{C} = \bigoplus_{j=1}^s \bigoplus_{k \geq 1} (\Lambda / (t - z_j)^k \Lambda)^{e_k(z_j)},$$

where the sum over  $k$  is finite,  $z_1, \dots, z_s$  are distinct elements in  $\mathbb{C} \setminus \{0, 1\}$ , and  $e_k(z_j) \geq 1$ , for all  $k$  and  $j$ .

It follows from (18) and (19) that the Alexander polynomial factors as

$$(20) \quad \Delta^G \doteq_{\mathbb{C}} (t - z_1)^{\mu_1} \cdots (t - z_s)^{\mu_s},$$

with

$$(21) \quad \mu_j = \sum_{k \geq 1} k e_k(z_j).$$

Every complex number  $z \neq 0$  defines a character  $z \in \mathbb{T}_G^0$ , and a local system  $\mathbb{C}_z$ . The next result relates the twisted Betti ranks corresponding to the roots  $z_j$  of the Alexander polynomial to the exponents  $e_k(z_j)$  appearing in the decomposition (19) of the complexified Alexander invariant.

**Proposition 6.13.** *Let  $G$  be a group with  $b_1(G) = 1$  and  $\Delta^G \neq \text{const}$ . Then, for  $j = 1, \dots, s$ ,*

$$b_1(G, z_j) = \mu_j \iff e_k(z_j) = 0, \quad \forall k > 1.$$

*Proof.* Since  $\Lambda$  is a PID and  $z_j \neq 1$ , we have

$$b_1(G, z_j) = \dim_{\mathbb{C}}(\mathbb{C}_{z_j} \otimes_{\Lambda} B_G \otimes \mathbb{C}) \sum_{k \geq 1} e_k(z_j).$$

Comparing this equality with (21) establishes the claim.  $\square$

The conditions  $e_k(z_j) = 0$  for  $k > 1$  represent a delicate restriction on the size of the Jordan blocks of a presentation matrix for the Alexander invariant. These conditions are not always satisfied, even for knots in  $S^3$ .

**Example 6.14.** By a classical result of Seifert (see [20, Theorem 7C.5]), given any Laurent polynomial  $f \in \Lambda_1 = \mathbb{Z}[t^{\pm 1}]$  with  $f(1) = \pm 1$  and  $f(t) \doteq f(t^{-1})$ , there exists a knot in  $S^3$  for which the Alexander invariant of the complement is isomorphic to  $\Lambda_1/(f)$ .

So fix such a polynomial, say  $f(t) = t^2 - t + 1$ , and for each  $k > 1$ , construct a knot  $K \subset S^3$  with group  $G = \pi_1(S^3 \setminus K)$  and Alexander invariant  $B_G = \Lambda_1/(f^k)$ . Clearly, all roots  $z_j$  of  $\Delta^G = f^k$  are multiple roots, and so  $b_1(G, z_j) < \mu_j$ .

**6.6. Fibrations over the circle.** Let  $M$  be a compact, connected manifold (possibly with boundary), which admits a locally trivial fibration over the circle,  $F \hookrightarrow M \xrightarrow{p} S^1$ . Let  $G = \pi_1(M)$ , and denote by  $h: F \rightarrow F$  the monodromy of the fibration.

Suppose the matrix of  $h_*: H_1(F; \mathbb{C}) \rightarrow H_1(F; \mathbb{C})$  does not have 1 as an eigenvalue. By the Wang sequence of the fibration, the map  $p_*: H_1(M; \mathbb{C}) \rightarrow H_1(S^1; \mathbb{C})$  is an isomorphism, and so  $b_1(G) = 1$ . If  $F$  is connected, the Alexander polynomial of  $G$  equals (up to units in  $\Lambda$ ) the characteristic polynomial of  $h_*$ ; in particular,  $\Delta^G \neq \text{const}$ , when  $b_1(F) > 0$ . Hence, Proposition 6.13 applies, and we obtain the following corollary.

**Corollary 6.15.** *In the above setup, write  $\Delta^G \doteq_{\mathbb{C}} \prod (t - z_j)^{\mu_j}$ , as in (20). The following are equivalent:*

- (1)  $b_1(G, z_j) = \mu_j$ , for all  $j$ .
- (2) The algebraic monodromy,  $h_*: H_1(F; \mathbb{C}) \rightarrow H_1(F; \mathbb{C})$ , is semisimple.

*Proof.* Since  $p_*: H_1(M; \mathbb{Z}) \rightarrow H_1(S^1; \mathbb{Z})$  is onto,  $h_*$  may be read off from (19).  $\square$

**Example 6.16.** As is well-known, any symplectic matrix  $A \in \mathrm{Sp}(2g, \mathbb{Z})$  may be realized as  $h_*: H_1(F; \mathbb{Z}) \rightarrow H_1(F; \mathbb{Z})$ , for some homeomorphism  $h: F \rightarrow F$  of a compact, orientable Riemann surface  $F$  of genus  $g$ , see [16, Theorem N13]. The mapping torus of this homeomorphism, call it  $M$ , fibers over the circle, with monodromy  $h$ . If 1 is not an eigenvalue of  $A$ , we are in the setup from Corollary 6.15. If, moreover,  $A$  is not diagonalizable, then the upper bounds on the twisted Betti numbers of  $G = \pi_1(M)$  are not attained.

As a concrete example, take  $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ . Then  $M$  is a torus bundle over the circle, with fundamental group

$$G = \langle x_1, x_2, x_3 \mid x_1x_2 = x_2x_1, x_3^{-1}x_1x_3 = x_1^{-1}, x_3^{-1}x_2x_3 = x_2x_1^{-1} \rangle$$

and Alexander polynomial  $\Delta^G = (1+t)^2$ . For the local system defined by the eigenvalue  $z_1 = -1$ , we have  $b_1(G, z_1) = 1$ , which is less than the multiplicity  $\mu_1 = 2$ .

## 7. SEIFERT LINKS

In this section, we examine the class of Seifert links, considered by Eisenbud and Neumann in [10]. For such links, the Alexander polynomial of the link group  $G$  determines all the characteristic varieties  $\mathcal{V}_k(G)$ ,  $k \geq 1$ .

**7.1. Links in homology 3-spheres.** Let  $\Sigma^3$  be a compact, smooth 3-manifold with  $H_*(\Sigma^3; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$ . A  $q$ -component link in  $\Sigma^3$  is a collection of disjoint circles,  $S_1, \dots, S_q$  ( $q \geq 1$ ), smoothly embedded in  $\Sigma^3$ . We shall fix orientations on both  $\Sigma^3$  and the link components  $S_i$ , and denote the resulting oriented link by

$$L = (\Sigma, S_1 \cup \dots \cup S_q).$$

The link exterior is the closed 3-manifold  $M_L = \Sigma^3 \setminus N_L$ , where  $N_L$  is an open tubular neighborhood of  $L$ . Clearly,  $M_L$  has the same homotopy type as the link complement,  $\Sigma^3 \setminus L$ . The link group,  $G_L := \pi_1(M_L)$ , is a finitely presented group, of positive deficiency. Its abelianization,  $(G_L)_{\mathrm{ab}} = \mathbb{Z}^q$ , comes endowed with a canonical basis, given by oriented meridians around the link components,  $e_1, \dots, e_q$ ; in particular,  $\mathbb{T}_{G_L} = \mathbb{T}_{G_L}^0$ . By either Lemma 6.2 or Lemma 6.3, the Alexander ideal of  $G_L$  is almost principal.

**7.2. The Eisenbud-Neumann calculus.** If the link exterior  $M_L$  admits a Seifert fibration, then  $L$  is called a *Seifert link*. Such links are completely classified. In this subsection, we consider only *positive* Seifert links, i.e., links of the form  $L = (\Sigma(k_1, \dots, k_n), S_1 \cup \dots \cup S_q)$ , with  $k_j \geq 1$  pairwise coprime integers,  $n \geq 3$  and  $n \geq q \geq 1$ , in the notation from [10, Proposition 7.3]. Such links are conveniently represented by ‘splice diagrams’. The diagram of the above link  $L$  has one positive *node*, connected to *arrowhead vertices*  $\{v_1, \dots, v_q\}$  and *boundary vertices*  $\{v_{q+1}, \dots, v_n\}$ , with the vertices  $\{v_1, \dots, v_n\}$  being labeled by the integer weights  $\{k_1, \dots, k_n\}$ ; see [10, p. 69].

This restriction is motivated by the fact that fundamental groups of non-positive Seifert links are very simple: they are either free groups, or groups of the form  $\mathbb{Z} \times F_{n-1}$ ,

with  $n \geq 2$ ; see [10, Proposition 7.3]. In all these cases, the Alexander polynomial has a single essential variable.

Theorem 12.1 from [10] allows one to read off the Alexander polynomial  $\Delta^L := \Delta^{G_L}$  directly from the splice diagram. We assume for simplicity that  $q \geq 2$ . Then

$$(22) \quad \Delta^L(t_1, \dots, t_q) = \frac{[(t_1^{N_1} \dots t_q^{N_q})^{N'} - 1]^{n-2}}{\prod_{j=q+1}^n [(t_1^{N_1} \dots t_q^{N_q})^{N'_j} - 1]},$$

where  $N = k_1 \dots k_q$ ,  $N_j = N/k_j$  for  $1 \leq j \leq q$ ,  $N' = k_{q+1} \dots k_n$ , and  $N'_j = N'/k_j$  for  $q+1 \leq j \leq n$ . We can assume that for  $j > q$ , we have  $k_j = 1$  iff  $j > q+s$ , with  $s$  a positive integer. Consequently,  $N'_j = N'$  for  $j > q+s$ . After simplification, formula (22) becomes

$$(23) \quad \Delta^L(t_1, \dots, t_q) = \frac{[(t_1^{N_1} \dots t_q^{N_q})^{N'} - 1]^{q+s-2}}{\prod_{j=q+1}^{q+s} [(t_1^{N_1} \dots t_q^{N_q})^{N'_j} - 1]}.$$

Since the integers  $k_j$  are pairwise coprime, it follows that  $\gcd(N_1, \dots, N_q) = 1$ , and hence the equation

$$(24) \quad t_1^{N_1} \dots t_q^{N_q} = 1$$

defines a subtorus  $\mathbb{T}' \subset \mathbb{T} := (\mathbb{C}^*)^q$ . In particular,  $\Delta^L$  has a single essential variable, namely  $u = t_1^{N_1} \dots t_q^{N_q}$ , and so Corollary 6.10 applies.

**7.3. Seifert links and quasi-projectivity.** It follows from the preceding subsections that Corollary 6.10 actually applies to all Seifert links with non-constant Alexander polynomial. Now we turn to a discussion of some analytic aspects of Seifert links.

Let  $(X, 0)$  be a complex quasi-homogeneous normal surface singularity. Then the surface  $X^* = X \setminus \{0\}$  is smooth, and admits a  $\mathbb{C}^*$ -action with finite isotropy groups  $\mathbb{C}_x^*$ . These isotropy groups are trivial, except for those corresponding to finitely many orbits associated to some points  $p_1, \dots, p_s$  in  $X^*$ ; let  $k_j$  be the order of  $\mathbb{C}_{p_j}^*$ .

The quotient  $X^*/\mathbb{C}^*$  is a smooth projective curve. For any finite subset  $B \subset X^*/\mathbb{C}^*$ , there is a surjective map  $f: M \rightarrow S$ , induced by the quotient map  $f_0: X^* \rightarrow X^*/\mathbb{C}^*$ , where  $S$  is the complement of  $B$  in  $X^*/\mathbb{C}^*$ , and  $M = f_0^{-1}(S)$ . Note that  $f_*: H_1(M) \rightarrow H_1(S)$  is surjective, since the fibers of  $f$  are connected.

In addition, the curve  $X^*/\mathbb{C}^*$  is rational if and only if the link  $L(X)$  of the singularity  $(X, 0)$  is a  $\mathbb{Q}$ -homology sphere (use Corollary (3.7) on p. 53 and Theorem (4.21) on p. 66 in [7]). In particular, if the link  $L(X)$  of the singularity  $(X, 0)$  is a  $\mathbb{Z}$ -homology sphere, then  $H_1(M) = \mathbb{Z}^q$  where  $q = |B|$ , and a basis is provided by small loops  $\gamma_b$  around the fiber  $F_b = f_0^{-1}(b)$  for  $b \in B$ . We have  $f_*(\gamma_b) = k_b \delta_b$ , with  $k_b$  the order of the isotropy groups of points  $x$  such that  $f_0(x) = b$ , and  $\delta_b$  a small loop around  $b \in \mathbb{P}^1$ . The set of critical values of the map  $f_0: X^* \rightarrow X^*/\mathbb{C}^*$  is exactly  $B_0 = \{y_1, \dots, y_s\}$ , with  $y_j = f_0(p_j)$ , and each fiber  $F_j = f_0^{-1}(y_j)$  is smooth (isomorphic to  $\mathbb{C}^*$ ), but of multiplicity  $k_j > 1$ . Writing down the map  $f_{0*}$  and using its surjectivity, we deduce that the integers  $k_1, \dots, k_s$  are pairwise coprime.

**Remark 7.1.** Let  $(X, 0)$  be the germ of an isolated complex surface singularity, such that the corresponding link  $L_X$  is an integral homology sphere. Let  $(Y, 0)$  be a curve singularity on  $(X, 0)$ . Then using the conic structure of analytic sets, we see that the local complement  $X \setminus Y$ , with  $X$  and  $Y$  Milnor representatives of the singularities  $(X, 0)$  and  $(Y, 0)$ , respectively, has the same homotopy type as the link complement  $M = L_X \setminus L_Y$ , where  $L_Y$  denotes the link of  $Y$ .

Moreover, if  $(X, 0)$  and  $(Y, 0)$  are quasi-homogeneous singularities at the origin of some affine space  $\mathbb{C}^N$ , with respect to the same weights, then the local complement can be globalized, i.e., replaced by the smooth quasi-projective variety  $X \setminus Y$ , where  $X$  and  $Y$  are this time affine varieties representing the germs  $(X, 0)$  and  $(Y, 0)$ , respectively.

A description along these lines of all Seifert links  $L$  is given in [10], p. 62. It follows that the link group  $G_L$  is quasi-projective.

**7.4. Characteristic varieties of Seifert links.** We are now in position to describe all characteristic varieties of Seifert links, solely in terms of their Alexander polynomials.

**Theorem 7.2.** *Let  $G_L$  be the group of the Seifert link  $L = (\Sigma(k_1, \dots, k_n), S_1 \cup \dots \cup S_q)$ , with  $k_i \geq 0$ ,  $n \geq 3$  and  $n \geq q \geq 1$ . Assume  $\Delta^L \neq \text{const}$ , that is,  $G_L$  is not a free group or  $\mathbb{Z}^2$ . Let  $D = m_1 D_1 + \dots + m_p D_p$  ( $m_j > 0$ ) be the effective divisor defined by  $\Delta^L = 0$  in  $\mathbb{T}_{G_L}$ . Then, for all  $j = 1, \dots, p$ , and all non-trivial characters  $\rho \in D_j$ ,*

$$\dim_{\mathbb{C}} H_1(G_L; \mathbb{C}_\rho) = m_j.$$

*Proof.* Since  $H_1(G; \mathbb{C}_\rho)$  and  $H^1(G; \mathbb{C}_\rho)$  are dual vector spaces, with  $\mathbb{C}_\rho$  viewed as a right  $\mathbb{C}G$ -module for homology and as a left  $\mathbb{C}G$ -module for cohomology, we may freely switch from homology to cohomology and back.

In the case of non-positive Seifert links, we only need to examine the groups from Example 3.3, with  $n \geq 3$ . This was done in Example 6.9. Exterior of positive Seifert knots fiber over  $S^1$ , with connected fibers and with finite-order geometric monodromy  $h$ ; see [10, Lemma 11.4]. In this case, the result follows from Corollary 6.15. Thus, we may suppose from now on that  $k_i \geq 1$ , for all  $i$ , and  $q \geq 2$ .

Using the analytic description of the Seifert link  $L = (\Sigma(k_1, \dots, k_n), S_1 \cup \dots \cup S_q)$  recalled in Remark 7.1 and the notation introduced in Section 7.3, we see that the link complement,  $M(L) = \Sigma(k_1, \dots, k_n) \setminus \{S_1 \cup \dots \cup S_q\}$ , has the homotopy type of the surface  $M$  obtained from the surface singularity  $X$  by deleting the orbits (regular for  $k_j = 1$  and singular for  $k_j > 1$ ) corresponding to the knots  $S_1, \dots, S_q$ . In other words, we have a finite set  $B \subset \mathbb{P}^1$  with  $|B| = q$  and an admissible mapping  $f: M \rightarrow S = \mathbb{P}^1 \setminus B$ , exactly as in §7.3.

Using the basis  $\{\gamma_b\}_{b \in B}$  of  $H_1(M)$ —respectively, the system of generators  $\{\delta_b\}_{b \in B}$ —described in §7.3, it follows that  $f$  induces an embedding of tori,  $f^*: \mathbb{T}(S) \rightarrow \mathbb{T}(M)$ , whose image is exactly the subtorus  $\mathbb{T}'$  defined in (24). Thus, we have an isomorphism

$$\mathbb{T}(M)/f^*(\mathbb{T}(S)) \xrightarrow{\cong} \mathbb{C}^*,$$

given by  $(t_1, \dots, t_q) \mapsto t_1^{N_1} \dots t_q^{N_q}$ . On the other hand, let  $i: F \rightarrow M$  be the inclusion of a generic (i.e., non-multiple) fiber of  $f$ . We know that  $F = \mathbb{C}^*$ , hence we get a restriction

morphism

$$i^*: \mathbb{T}(M) \rightarrow \mathbb{T}(F) = \mathbb{C}^*.$$

One can calculate the image of  $i_*: H_1(F) \rightarrow H_1(M)$ , as follows. The set of critical values  $C(f)$  of  $f: M \rightarrow S$  consists precisely of the  $s$  points in  $S$  corresponding to the knots  $S_j$  (which are not used in the link, i.e., for  $j > q$ ) with multiplicity  $k_j > 1$ . Then, if we delete these critical values and the corresponding multiple fibers, we obtain a locally trivial fibration  $f': M' \rightarrow S'$ . If  $i': F \rightarrow M'$  denotes the inclusion, it is easy to determine  $\text{im}(i'_*) = \ker(f'_*)$  and then one obtains  $\text{im}(i_*)$  using the natural projection  $H_1(M') \rightarrow H_1(M)$  (see [8] for a similar computation). If  $\sigma \in H_1(F)$  denotes the natural generator, it follows that  $i_*(\sigma) = N'(N_1\gamma_1 + \cdots + N_q\gamma_q)$ . Hence,

$$(25) \quad i^*(t_1, \dots, t_q) = (t_1^{N_1} \cdots t_q^{N_q})^{N'}.$$

Using [8], it follows that the irreducible components of  $\mathcal{V}_1(M) = \mathcal{V}_1(G_L)$  associated to the admissible map  $f: M \rightarrow S$  are parametrized by

$$\widehat{T}(f) = \frac{\ker i^*}{f^*(\mathbb{T}(S))} = \Gamma_{N'},$$

for  $q > 2$ , respectively by  $\Gamma_{N'} \setminus \{1\}$ , for  $q = 2$ , where  $\Gamma_{N'}$  is the cyclic group of roots of unity of order  $N'$ . By virtue of (23) and Corollary 3.2(2), these components coincide with the irreducible components of  $V(\Delta^L)$ .

Now, for a character  $\rho \in \widehat{T}(f)$ , it follows from [8] that

$$(26) \quad \dim H^1(M; \mathbb{C}_{\rho'}) \geq -\chi(S) + |\Sigma(R^0 f_*(\mathbb{C}_{\rho}))|,$$

for any  $\rho'$  in the corresponding irreducible component  $W_{f,\rho}$ , provided the right hand side is strictly positive. Moreover, equality holds with finitely many exceptions.

Using formula (23) and Corollary 6.10, we obtain

$$(27) \quad \dim H^1(M; \mathbb{C}_{\rho'}) \leq q + s - 2 - |I(\alpha(\rho))|,$$

for  $\rho' \neq 1$ , where  $\alpha(\rho) = \rho_1^{N_1} \cdots \rho_q^{N_q}$  and  $I(\alpha) = \{j \mid q < j \leq q + s \text{ and } \alpha^{N'_j} = 1\}$ .

On the other hand,  $\chi(S) = 2 - q$ . If  $c \in C(f)$  and  $T(F_c) = f^{-1}(D_c)$  is a small tubular neighborhood of the multiple fiber  $F_c$ , with  $D_c$  a small open disk centered at  $c$ , then the inclusion  $F_c \rightarrow T(F_c)$  is a homotopy equivalence; hence,  $H_1(F_c) = H_1(T(F_c))$ . The inclusion  $F \rightarrow T(F_c)$  induces then a morphism  $H_1(F) \rightarrow H_1(T(F_c))$  which is just multiplication by  $k_c: \mathbb{Z} \rightarrow \mathbb{Z}$ . Combining this fact with formula (25), it follows that the inclusion  $i_c: F_c \rightarrow M$  induces a morphism  $i_c^*: \mathbb{T}(M) \rightarrow \mathbb{T}(F_c) = \mathbb{C}^*$  given by

$$(28) \quad i_c^*(t_1, \dots, t_q) = (t_1^{N_1} \cdots t_q^{N_q})^{N'_c}.$$

Now recall from [8] that  $c \in \Sigma(R^0 f_*(\mathbb{C}_{\rho}))$  if and only if  $H^0(T(F_c); \mathbb{C}_{\rho}) = 0$ . This happens exactly when the restriction of the local system  $\mathbb{C}_{\rho}$  to  $T(F_c)$ —or, equivalently, to  $F_c$ —is non-trivial. By formula (28), this is the same as  $c \notin I(\alpha(\rho))$ , i.e.,

$$(29) \quad |\Sigma(R^0 f_*(\mathbb{C}_{\rho}))| = s - |I(\alpha(\rho))|.$$

This establishes the reverse inequality in (27). Since the right hand side from (27) is precisely the multiplicity from the assertion to be proved, we are done.  $\square$

**Example 7.3.** Let  $p$  and  $q$  be two positive integers such that  $(p, q) = 1$  and let  $n \geq 2$ . Let  $L$  be the algebraic link in  $S^3$  associated to the plane curve singularity

$$(C, 0): x^{pn} - y^{qn} = 0.$$

In terms of Seifert's notation, this is the link

$$L = (\Sigma(1, \dots, 1, p, q), S_1 \cup \dots \cup S_n),$$

with 1 occurring  $n$  times. Formula (22) becomes

$$\Delta^L(t_1, \dots, t_n) = \frac{[(t_1 \cdots t_n)^{pq} - 1]^n}{[(t_1 \cdots t_n)^p - 1][(t_1 \cdots t_n)^q - 1]}.$$

The irreducible components of the divisor  $D$  in  $\mathbb{T}^n$  defined by  $\Delta^L(t_1, \dots, t_n) = 0$  are parametrized by the elements of the group  $\Gamma_p \times \Gamma_q$ , for  $n \geq 3$ , respectively by the nontrivial elements, for  $n = 2$ . (Here  $\Gamma_k$  denotes the multiplicative group of  $k$ -th roots of unity.) More precisely, for  $(a, b) \in \Gamma_p \times \Gamma_q$ , the corresponding irreducible component

$$(30) \quad D_{a,b}: t_1 \cdots t_n - a \cdot b = 0$$

(a subtorus through 1 if  $a \cdot b = 1$ , and a translated subtorus if  $a \cdot b \neq 1$ ) occurs with multiplicity  $m(a, b) = n - 2 + |\{a - 1, b - 1\} \cap \mathbb{C}^*|$ .

It follows that the characteristic variety  $\mathcal{V}_1(G_L)$  has irreducible components given by (30), with  $(a, b) \in \Gamma_p \times \Gamma_q$ , for  $n \geq 3$ ; if  $n = 2$ , one needs to replace  $D_{1,1}$  by  $\{1\}$ .

**Acknowledgment.** This work was started while the authors visited the Abdus Salam International Centre for Theoretical Physics in Fall, 2006. We thank ICTP for its support and excellent facilities, and Lê Dũng Tráng for a helpful hint.

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