Advanced Studies in Pure Mathematics 27, 2000
Arrangements - Tokyo 1998
pp. 185–215

Cohomology rings and nilpotent quotients of real and complex arrangements

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Dedicated to Peter Orlik on his 60th birthday

Abstract.

For an arrangement with complement X and fundamental group G, we relate the truncated cohomology ring, $H^{\leq 2}(X)$, to the second nilpotent quotient, G/G_3. We define invariants of G/G_3 by counting normal subgroups of a fixed prime index p, according to their abelianization. We show how to compute this distribution from the resonance varieties of the Orlik-Solomon algebra mod p. As an application, we establish the cohomology classification of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$.

§0. Introduction

1. Two basic homotopy-type invariants of a path-connected space X are: the cohomology ring, $H^*(X)$, and the fundamental group, $G = \pi_1(X)$. Given X and X', one would like to know:

(I) Is there an isomorphism $H^{\leq q}(X) \cong H^{\leq q}(X')$ between the cohomology rings, up to degree q?

(II) Is there an isomorphism $G/G_{q+1} \cong G'/G'_{q+1}$ between the qth nilpotent quotients?

We single out a class of spaces—including complements of complex hyperplane arrangements, complements of 'rigid' links, and complements of arrangements of transverse planes in $\mathbb{R}^4$—for which the above questions are amenable to a detailed study, capable of yielding classification results. The invariants that we use have a dual nature, being able to capture both the ring-theoretic properties of the cohomology of X, and the group-theoretic properties of the nilpotent quotients of G. Our main

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1991 *Mathematics Subject Classification.* Primary 52B30, 57M05; Secondary 20F14, 20J05.

Key words and phrases. complex hyperplane arrangement, 2-arrangement, cohomology ring, resonance variety, nilpotent quotient, prime index subgroup.

result is an explicit correspondence between two sets of invariants—one determined by the vanishing cup products in $H^{\leq 2}(X)$, the other by the finite-index subgroups of G/G_3.

2. For $q = 1$, questions (I) and (II) are equivalent, provided H_1 is torsion free. Indeed, $H^1(X) = G/G_2$ under that assumption. For $q = 2$, the two questions are still equivalent, under some additional conditions: If H_2 is also torsion-free, and the cup-product map $\mu : H^1 \wedge H^1 \to H^2$ is surjective, then:

$$H^{\leq 2}(X) \cong H^{\leq 2}(X') \text{ if and only if } G/G_3 \cong G'/G'_3.$$

Section 1 is devoted to a proof of this fact. A key ingredient is the vanishing of the Hurewicz map $\pi_2(X) \to H_2(X)$, which permits us to identify $H^{\leq 2}(X)$ with $H^{\leq 2}(G)$. The other ingredient is the interpretation of the k-invariant of the extension $0 \to G_2/G_3 \to G/G_3 \to G/G_2 \to 0$, in terms of the cup-products of G.

In Section 2, we use commutator calculus to describe the nilpotent quotients of G. We restrict our attention to spaces X, for which $G = \pi_1(X)$ has a finite presentation $G = \mathbb{F}/R$, with $R \subset [\mathbb{F}, \mathbb{F}]$. The cup products in $H^{\leq 2}(G)$ can then be computed from the second order Fox derivatives of the relators.

3. The invariants of the cohomology ring that we use are the resonance varieties, first introduced by Falk [11] in the context of complex hyperplane arrangements. The dth resonance variety of X, with coefficients in a field $\mathbb{K}$, is the set $\mathcal{R}_d(X, \mathbb{K})$ of cohomology classes $\lambda \in H^1(X, \mathbb{K})$ for which there is a subspace $W \subset H^1(X, \mathbb{K})$, of dimension $d+1$, such that $\mu(\lambda \wedge W) = 0$.

In Section 3, we prove that $\mathcal{R}_d(X, \mathbb{K})$ equals $\mathcal{R}_d(G, \mathbb{K})$, the resonance variety of the Eilenberg-MacLane space $K(G, 1)$. Moreover, we exploit the Fox calculus interpretation of cup products to show that the varieties $\mathcal{R}_d(G, \mathbb{K})$ are the determinantal varieties of the 'linearized' Alexander matrix of G.

4. A well-known invariant of a group G is the number of normal subgroups of fixed prime index. For a commutator-relators group, that number depends only on $n = \operatorname{rank} G/G_2$, and the prime p. In order to get a finer invariant, we consider the distribution of index p subgroups, according to their abelianization. The ν-invariants of the nilpotent quotients G/G_{q+1} are defined as follows:

$$\nu_{p,d}(G/G_{q+1}) = \# \left\{ K \lhd G/G_{q+1} \;\middle|\; \begin{array}{l} [G/G_{q+1} : K] = p \text{ and} \\ \dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{Z}_p = d \end{array} \right\}.$$

In Section 4, we show how to compute the ν-invariants of G/G_3 from the stratification of $\mathbb{P}(\mathbb{Z}_p^n)$ by the projectivized $\mathbb{Z}_p$-resonance varieties of X:

$$\nu_{p,d}(G/G_3) = \#(\mathcal{P}_d(X, \mathbb{Z}_p) \setminus \mathcal{P}_{d+1}(X, \mathbb{Z}_p)).$$

This formula makes the computation of the ν-invariants practical. It also makes clear that the mod p resonance varieties of X, which are defined solely in terms of $H^{\leq 2}(X)$, do capture significant group-theoretic information about G/G_3.

5. In the case where X is the complement of a complex hyperplane arrangement, the varieties $\mathcal{R}_d(X, \mathbb{C})$ have been extensively studied by Falk, Yuzvinsky, Libgober, Cohen, and Suciu [11, 33, 19, 20, 7]. The top variety, $\mathcal{R}_1(X, \mathbb{C})$, is a complete invariant of the cohomology ring $H^{\leq 2}(X)$. Moreover, $\mathcal{R}_1(X, \mathbb{C})$ is a union of linear subspaces intersecting only at the origin, and $\mathcal{R}_d(X, \mathbb{C})$ is the union of those subspaces of dimension at least $d + 1$.

In Section 5, we use these results to derive a simple consequence. Since $\mathcal{R}_d(X, \mathbb{C})$ has integral equations, we may consider its reduction mod p. If that variety coincides with $\mathcal{R}_d(X, \mathbb{Z}_p)$, we have:

$$\nu_{p,d-1}(G/G_3) = \frac{p^d - 1}{p - 1} m_d,$$

where m_d is the number of components of $\mathcal{R}_1(X, \mathbb{C})$ of dimension d.

In general, though, this formula fails, due to a different resonance at 'exceptional' primes. For such primes p, the variety $\mathcal{R}_d(X, \mathbb{Z}_p)$ is not necessarily the union of the components of $\mathcal{R}_1(X, \mathbb{Z}_p)$ of dimension $\geq d + 1$. Furthermore, $\mathcal{R}_1(X, \mathbb{C})$ mod p and $\mathcal{R}_1(X, \mathbb{Z}_p)$ may differ in the number of non-local components, as well as in the dimensions of those components. Most strikingly, $\mathcal{R}_1(X, \mathbb{Z}_p)$ may have non-local components, even though $\mathcal{R}_1(X, \mathbb{C})$ mod p has none.

6. Much of the original motivation for this paper came from an effort to understand Ziegler's pair of arrangements of 4 transverse planes in $\mathbb{R}^4$. Those arrangements have isomorphic lattices, but their complements have non-isomorphic cohomology rings, see [34]. In an earlier work [23], we investigated the homotopy types of complements of 2-arrangements, obtaining a complete classification for $n \leq 6$ planes. This left open the problem of classifying cohomology rings for $n > 4$.

In Section 6, we start a study of the varieties $\mathcal{R}_d(X, \mathbb{C})$, where X is the complement of a 2-arrangement. The resonance varieties of real arrangements exhibit a much richer geometry than those of complex

arrangements. Most strikingly, $\mathcal{R}_1(X,\mathbb{C})$ may not be a union of linear subspaces, and it may not determine $H^*(X)$.

Using the ν-invariants of G/G_3, we establish the cohomology classification of complements of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$. With one exception, this classification coincides with the homotopy-type classification from [23]. The exception is Mazurovskiĭ's pair [24]. The two complements, X and X', have isomorphic cohomology rings, and thus $G/G_3 \cong G'/G'_3$. On the other hand, $\nu_{3,2}(G/G_4) = 162$ and $\nu_{3,2}(G'/G'_4) = 172$.

As this example shows, the ν-invariants of the third nilpotent quotient *cannot* be computed from the resonance varieties of the cohomology ring. To arrive at a more conceptual understanding of these invariants, one needs to look beyond cup-products, and on to the Massey products. This will be pursued elsewhere.

Acknowledgment

We wish to thank Sergey Yuzvinsky for valuable discussions regarding the material in Section 5, and the referee for carefully reading the manuscript. The computations for this work were done with the help of the packages *GAP 3.4.4* ([30]), *Macaulay 2* ([14]), and *Mathematica 3.0*.

Contents

§1. Cohomology ring and second nilpotent quotient

In this section, we introduce a class of spaces that abstract the cohomological essence of hyperplane arrangements. We then relate the cohomology ring of such a space X to the second nilpotent quotient of the fundamental group of X.

1.1. Cohomology ring

All the spaces considered in this paper have the homotopy type of a connected CW-complex with finite 2-skeleton. Let X be such a space. Consider the following conditions on the cohomology ring of X:

(A) The homology groups $H_*(X)$ are free abelian.
(B) The cup-product map $\mu_X : \bigwedge^* H^1(X) \to H^*(X)$ is surjective.

If conditions (A) and (B) only hold for $1 \leq * \leq n$, we will refer to them as (A_n) and (B_n).

Example 1.2. The basic example we have in mind is that of the complement, $X = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$, of a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$. As shown by Brieskorn [3] (solving a conjecture of Arnol'd), such a space X satisfies conditions (A) and (B). Moreover, as shown by Orlik and Solomon [25], the *intersection lattice* of the arrangement, $L(\mathcal{A}) = \left\{\bigcap_{H\in\mathcal{B}} H \,|\mathcal{B} \subseteq \mathcal{A}\right\}$, determines the cohomology ring of X, as follows:

$$H^*(X) = \bigwedge^* \mathbb{Z}^n \Big/ \Big(\partial e_{\mathcal{B}} \;\Big|\; \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}|\Big).$$

Here $\bigwedge^* \mathbb{Z}^n$ is the exterior algebra on generators $e_1, \ldots, e_n$ dual to the meridians of the hyperplanes; and, if $\mathcal{B} = \{H_{i_1}, \ldots, H_{i_r}\}$ is a subarrangement, then $e_{\mathcal{B}} = e_{i_1} \cdots e_{i_r}$, and $\partial e_{\mathcal{B}} = \sum_q (-1)^q e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$. See [26] for a thorough treatment of hyperplane arrangements.

Let X be a space satisfying conditions (A_n) and (B_n). The first condition and the Universal Coefficient Theorem (see [2], Theorem 7.1, p. 281) imply that $H_*(X) = H^*(X)$, for $* \leq n$. Write $H = H_1(X) = H^1(X)$. Denote by I^* the kernel of the cup-product map. Condition (B_n) can be restated as saying that the following sequence is exact:

$$0 \to I^* \xrightarrow{\iota} \bigwedge^* H \xrightarrow{\mu_X} H^*(X) \to 0, \qquad \text{for } * \leq n. \tag{1}$$

By condition (A_n), this is in fact a split-exact sequence.

1.3. Hurewicz homomorphism

The following lemma was proved by Randell [27] in the case where X is the complement of a complex hyperplane arrangement.

Lemma 1.4. *If X satisfies conditions (A_n) and (B_n), then the Hurewicz homomorphism, $h : \pi_i(X) \to H_i(X)$, is the zero map, for $2 \leq i \leq n$.*

Proof. The proof is exactly as in [27]. Let $p : \widetilde{X} \to X$ be the universal covering map. Recall that $p_* : \pi_i(\widetilde{X}) \to \pi_i(X)$ is an isomorphism, for $i \geq 2$. By naturality of the Hurewicz map, universal coefficients, and condition (A_n), it is enough to show that $p^* : H^i(X) \to H^i(\widetilde{X})$ is the zero map. This follows from $H^1(\widetilde{X}) = 0$, condition (B_n), and the naturality of cup products: $p^* \circ \mu_X = \mu_{\widetilde{X}} \circ \wedge^i p^*$. Q.E.D.

1.5. Group cohomology

Let G be a group. The (co)homology groups of G are by definition those of the corresponding Eilenberg-MacLane space: $H_*(G) = H_*(K(G,1))$ and $H^*(G) = H^*(K(G,1))$. Consider the following homological conditions on G:

(A′) The homology groups $H_1(G)$ and $H_2(G)$ are finitely generated free abelian.

(B′) The cup-product map $\mu_G : H^1(G) \wedge H^1(G) \to H^2(G)$ is surjective.

Proposition 1.6. *Let X be a space satisfying conditions* (A$_2$) *and* (B$_2$)*, and let $G = \pi_1(X)$ be its fundamental group. Then the following hold*:

(a) *$H_1(G) \cong H_1(X)$ and $H_2(G) \cong H_2(X)$.*

(b) *The rings $H^{\leq 2}(G)$ and $H^{\leq 2}(X)$ are isomorphic.*

Therefore, G satisfies conditions (A′) *and* (B′).

Proof. Recall X has the homotopy type of a connected CW-complex Y with finite 2-skeleton. A $K(G,1)$ space may be obtained from Y by attaching suitable cells of dimension ≥ 3. The resulting map, $j : X \to K(G,1)$, induces an isomorphism $H_1(X) \cong H_1(G)$. From the Hopf exact sequence $\pi_2(X) \xrightarrow{h} H_2(X) \to H_2(G) \to 0$ and Lemma 1.4, we get $H_2(X) \cong H_2(G)$. This finishes the proof of (a).

By universal coefficients, the map $j^* : H^i(G) \to H^i(X)$ is a group isomorphism, for $i \leq 2$. By naturality of cup products, we have $j^*\mu_G(a \wedge b) = \mu_X(j^*a \wedge j^*b)$. This proves (b). Q.E.D.

Remark 1.7. The above conditions on X also appear in [1, 29]. The surjectivity of $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$ is stated there dually, as the injectivity of the holonomy map, $\mu^\top : H_2(X) \to \bigwedge^2 H_1(X)$.

1.8. Nilpotent quotients

Let G be a finitely generated group. The lower central series of G is defined inductively by $G_1 = G$, $G_{q+1} = [G, G_q]$, where $[G, G_q]$ denotes the subgroup of G generated by the commutators $[x,y] = xyx^{-1}y^{-1}$ with $x \in G$ and $y \in G_q$. The quotient G_q/G_{q+1} is a finitely generated abelian group, called the qth *lower central series quotient* of G. The quotient G/G_{q+1} is a nilpotent group, called the qth *nilpotent quotient* of G. See [21] for details.

We will be mainly interested in the second nilpotent quotient, G/G_3. This group is a central extension of finitely generated abelian groups,

$$0 \to G_2/G_3 \to G/G_3 \to G/G_2 \to 0. \tag{2}$$

The extension is classified by the k-invariant, $\bar{\chi} \in H^2(G/G_2; G_2/G_3)$. The isomorphism type of G/G_3 is determined by G/G_2, G_2/G_3, and $\bar{\chi}$, as follows.

Let G and G' be two groups. Then $G/G_3 \cong G'/G'_3$ if and only if there exist isomorphisms $\phi : G/G_2 \to G'/G'_2$ and $\psi : G_2/G_3 \to G'_2/G'_3$ under which the k-invariants correspond: $\psi_*(\bar{\chi}) = \phi^*(\bar{\chi}') \in H^2(G/G_2; G'_2/G'_3)$.

Now suppose $H = G/G_2$ is torsion-free. As is well-known, $H_*(H) \cong \bigwedge^* H$. The *classifying map* for the extension (2),

$$\chi : \bigwedge^2 H \to G_2/G_3$$

is the image of $\bar{\chi}$ under the epimorphism

$$H^2(H; G_2/G_3) \to \mathrm{Hom}(\bigwedge^2 H, G_2/G_3)$$

provided by the Universal Coefficient Theorem (see [9]). It is given by $\chi(x \wedge y) = [x, y]$ (see [4], Exercise 8, p. 97). The condition that the k-invariants of G/G_3 and G'/G'_3 correspond translates to $\psi \circ \chi = \chi' \circ \wedge^2 \phi$. We shall write this equivalence relation between classifying maps as $\chi \sim \chi'$.

Suppose now that G_2/G_3 is also torsion-free. Then, the universal coefficient map is an isomorphism, and so $\bar{\chi}$ and χ determine each other. Thus, for a group G with G/G_2 and G_2/G_3 torsion-free, the isomorphism type of G/G_3 is completely determined by the equivalence class of the classifying map χ.

1.9. Cup product and commutators

The 5-term exact sequence for the extension $0 \to G_2 \to G \xrightarrow{\alpha} G/G_2 \to 0$ yields:

$$H_2(G) \xrightarrow{\alpha_*} H_2(G/G_2) \xrightarrow{\delta} G_2/G_3 \to 0. \tag{3}$$

Under the identification $H_2(G/G_2) \cong \bigwedge^2 H$, the boundary map δ corresponds to the classifying map χ (see [4], Exercise 6, p. 47). The next lemma interprets the map α_* in terms of the ring structure of $H^*(G)$.

Lemma 1.10. *The map $\alpha_* : H_2(G) \to \bigwedge^2 H$ is the dual of the cup-product map $\mu_G : H^1(G) \wedge H^1(G) \to H^2(G)$.*

Proof. Follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(H)\wedge H^1(H) & \xrightarrow{\alpha^*\wedge\alpha^*} & H^1(G)\wedge H^1(G) \\ \Big\downarrow{\scriptstyle \mu_H} & & \Big\downarrow{\scriptstyle \mu_G} \\ H^2(H) & \xrightarrow{\alpha^*} & H^2(G) \end{array}$$

and the fact that the top and left arrows are isomorphisms. Q.E.D.

The following proposition generalizes a result proved by Massey and Traldi [22] in the case where G is a link group.

Proposition 1.11. *Let G be a group satisfying conditions* (A′) *and* (B′). *Then G_2/G_3 is torsion-free, and the following is a split exact sequence*:

$$0 \to H_2(G) \xrightarrow{\mu^\top} \bigwedge^2 H \xrightarrow{\chi} G_2/G_3 \to 0. \tag{4}$$

Proof. The proof follows closely that in [22]. By Lemma 1.10, sequence (3) can be written as $H_2(G) \xrightarrow{\mu^\top} \bigwedge^2 H \xrightarrow{\chi} G_2/G_3 \to 0$. By condition (B′), the map $\mu^\top$ is a monomorphism, whence the exactness of (4).

Since $\mu : \bigwedge^2 H \to H^2(G)$ is an epimorphism between finitely generated free abelian groups, it admits a splitting. Hence $\mu^\top$ is a split injection, and so $\chi^\top$ is a split surjection. Since $\bigwedge^2 H$ is torsion-free, G_2/G_3 is also torsion-free. Q.E.D.

Remark 1.12. The injectivity of $\mu^\top : H_2(G) \to \bigwedge^2 H$ is equivalent to the vanishing of $\Phi_3(G)$, where $H_2(G) = \Phi_2(G) \supset \Phi_3(G) \supset \cdots$ is the Dwyer filtration, $\Phi_k(G) = \ker(H_2(G) \to H_2(G/G_{k-1}))$, see [9].

1.13. Isomorphisms

The next result is an immediate consequence of Proposition 1.11:

Proposition 1.14. *Let X be a space satisfying conditions* (A_2) *and* (B_2), *and let $G = \pi_1(X)$. Then $I^2 = G_2/G_3$, and the exact sequence*

$$0 \to I^2 \xrightarrow{\iota} \bigwedge^2 H^1(X) \xrightarrow{\mu} H^2(X) \to 0 \tag{5}$$

is the dual of sequence (4).

We are now ready to establish the correspondence between the truncated cohomology ring of X and the second nilpotent quotient of

$G = \pi_1(X)$. A version of the equivalence $(b) \Leftrightarrow (c)$ below, with χ replaced by $\mu^\top$, was first established by Traldi and Sakuma [32], in the case where X is a link complement.

Theorem 1.15. *Let X and X' be two spaces satisfying conditions* (A_2) *and* (B_2)*, and let G and G' be their fundamental groups. The following are equivalent*:

(a) $H^*(X) \cong H^*(X')$ *for* $* \leq 2$*;*
(b) $G/G_3 \cong G'/G'_3$*;*
(c) $\chi \sim \chi'$.

Proof. $(a) \Leftrightarrow (c)$. By Proposition 1.14, sequence (5) is exact, and $\chi = \iota^\top$. The equivalence follows from the definitions.

$(b) \Leftrightarrow (c)$. By Propositions 1.6 and 1.14, the first two lower central series quotients of G and G' are torsion-free. The equivalence follows from the discussion in 1.8. Q.E.D.

1.16. Invariants of $H^{\leq 2}(X)$ and G/G_3

In view of Theorem 1.15, an invariant of either the truncated cohomology ring $H^{\leq 2}(X)$, or the second nilpotent quotient G/G_3, or the classifying map χ, is an invariant of the other two. We will define in subsequent sections a series of invariants of both $H^{\leq 2}(X)$ and G/G_3, and relate them one to another. For now, let us define invariants of χ, following an idea of Ziegler [34], that originated from Falk's work on minimal models of arrangements [10].

Let $\mu_H : \bigwedge^i H \otimes \bigwedge^j H \to \bigwedge^{i+j} H$ be the multiplication in the exterior algebra $\bigwedge^* H$. Consider the following finitely generated abelian group:

$$Z_{i,j}(\chi) = \operatorname{coker}\left(\bigwedge^i H \otimes \bigwedge^j G_2/G_3 \xrightarrow{\mathrm{id} \otimes \bigwedge^j \chi^\top} \bigwedge^i H \otimes \bigwedge^{2j} H \xrightarrow{\mu_H} \bigwedge^{i+2j} H\right).$$

Clearly, if $\chi \sim \chi'$ then $Z_{i,j}(\chi) \cong Z_{i,j}(\chi')$. Thus, the rank and elementary divisors of $Z_{i,j}(\chi)$ provide invariants of both $H^{\leq 2}(X)$ and G/G_3.

§2. Generators and relators

In this section, we write down explicitly some of the maps introduced in the previous section. We start with a review of some basic facts about Hall commutators and the Fox calculus.

2.1. Basic commutators

Let $\mathbb{F}(n)$ be the free group on generators $x_1, \ldots, x_n$. A *basic commutator* in $\mathbb{F} = \mathbb{F}(n)$ is defined inductively as follows (see [12, 21]):

(a) Each basic commutator c has length $\ell(c) \in \mathbb{N}$.
(b) The basic commutators of length 1 are the generators $x_1, \dots, x_n$; those of length > 1 are of the form $c = [c_1, c_2]$, where c_1, c_2 are previously defined commutators and $\ell(c) = \ell(c_1) + \ell(c_2)$.
(c) Basic commutators of the same length are ordered arbitrarily; if $\ell(c) > \ell(c')$, then $c > c'$.
(d) If $\ell(c) > 1$ and $c = [c_1, c_2]$, then $c_1 < c_2$; if $\ell(c) > 2$ and $c = [c_1, [c_2, c_3]]$, then $c_1 \geq c_2$.

The basic commutators of the form $c = [x_{i_1}, [x_{i_2}, [\dots [x_{i_{q-1}}, x_{i_q}] \dots]]]$ are called *simple*. We shall write them as $c = [x_{i_1}, x_{i_2}, \dots, x_{i_q}]$. For $q \leq 3$, all basic commutators are simple.

The following theorem of Hall is well-known (see *loc. cit.*):

Theorem 2.2. *The group $\mathbb{F}_q/\mathbb{F}_{q+1}$ is free abelian, and has a basis consisting of the basic commutators of length q.*

In particular, if $w \in \mathbb{F}$ and $c_1, \dots, c_r$ are the basic commutators of length $< q$, then $w^{(q)} := w \bmod \mathbb{F}_q$ may be written uniquely as $w^{(q)} = c_1^{e_1} c_2^{e_2} \cdots c_r^{e_r}$, for some integers $e_1, \dots, e_r$.

The Hall commutators may be used to write down presentations for the nilpotent quotients of a finitely presented group $G = \mathbb{F}/R$. Indeed, if $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, we have the following presentation for $G/G_q = \mathbb{F}/R\mathbb{F}_q$:

$$G/G_q = \langle x_1, \dots, x_n \mid r_1^{(q)}, \dots, r_m^{(q)}, c_1, \dots, c_l \rangle, \tag{6}$$

where $r_k^{(q)} = r_k \mod \mathbb{F}_q$, and $\{c_h\}_{1 \leq h \leq l}$ are the basic commutators of length q.

2.3. Fox calculus

Let $\mathbb{Z}\mathbb{F}$ be the group ring of $\mathbb{F}$, with augmentation map $\epsilon : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$ given by $\epsilon(x_i) = 1$. To each x_i there corresponds a *Fox derivative*, $\partial_i : \mathbb{Z}\mathbb{F} \to \mathbb{Z}\mathbb{F}$, given by $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}$ and $\partial_i(uv) = \partial_i(u)\epsilon(v) + u\partial_i(v)$. The higher Fox derivatives, $\partial_{i_1,\dots,i_k}$, are defined inductively in the obvious manner. The composition of the augmentation map with the higher derivatives yields operators $\epsilon_{i_1,\dots,i_k} := \epsilon \circ \partial_{i_1,\dots,i_k} : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$.

Let $\alpha : \mathbb{F}(n) \to \mathbb{Z}^n$ be the abelianization map, given by $\alpha(x_i) = t_i$. The following lemma is left as an exercise in the definitions.

Lemma 2.4. *We have*:
(a) $\partial_i[u, v] = (1 - uvu^{-1})\partial_i u + (u - [u, v])\partial_i v$.
(b) $\alpha(\partial_i[x_{i_1}, x_{i_2}, \dots, x_{i_q}]) = (t_{i_1} - 1) \cdots (t_{i_{q-2}} - 1)\big((t_{i_{q-1}} - 1)\delta_{i,i_q} - (t_{i_q} - 1)\delta_{i,i_{q-1}}\big)$.

(c) $\epsilon_I(w) = 0$, *if* $w \in \mathbb{F}_q$ *and* $|I| < q$.
(d) $\epsilon_I(uv) = \epsilon_I(u) + \epsilon_I(v)$, *if* $u, v \in \mathbb{F}_q$ *and* $|I| = q$.

2.5. Commutator relations

We now make more explicit some of the constructions from section 1, for the following class of groups.

Definition 2.6. A group G is called a *commutator-relators group* if it admits a presentation $G = \mathbb{F}(n)/R$, where R is the normal closure of a finite subset of $[\mathbb{F}, \mathbb{F}]$.

In other words, G has a finite presentation $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$, and $G/G_2 = \mathbb{Z}^n$. Commutator-relators groups appear as fundamental groups of certain spaces that we shall encounter later on. The following proposition gives sufficient conditions for this to happen.

Proposition 2.7. *Let X be a space that is homotopy equivalent to a finite CW-complex Y, with 1-skeleton $Y^{(1)} = \bigvee_{i=1}^n S_i^1$. If $H_1(X) = \mathbb{Z}^n$, then $G = \pi_1(X)$ is a commutator-relators group.*

Proof. The 2-skeleton $Y^{(2)} = \bigvee_{i=1}^n S_i^1 \cup \bigcup_{k=1}^m e_k^2$ determines a presentation $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$. A presentation matrix for the abelianization of G is $E = (\epsilon_i(r_k))$. Since $H_1(X) = \mathbb{Z}^n$, we have $H_1(G) = \mathbb{Z}^n$. Thus, E is equivalent to the zero matrix, and hence E *is* the zero matrix. Thus, all relators r_k are commutators. Q.E.D.

Now let $\phi : \mathbb{F} \to G$ be the quotient map, and let $\alpha : G \to G/G_2$ be the abelianization map. Set $t_i = \alpha(\phi(x_i))$. Then $\{t_1, \ldots, t_n\}$ form a basis for $H_1(G)$, and their Kronecker duals, $\{e_1, \ldots, e_n\}$, form a basis for $H^1(G)$.

By the Hopf formula, we have $H_2(G) = R/[R, \mathbb{F}]$. Assume that $H_2(G)$ is free abelian, and let $\theta_k = r_k \bmod [R, \mathbb{F}]$. Then $\{\theta_1, \ldots, \theta_m\}$ form a basis for $H_2(G)$, and their duals, $\{\gamma_1, \ldots, \gamma_m\}$, form a basis for $H^2(G)$.

Proposition 2.8. *Let G be a commutator-relators group, such that $H_2(G)$ is free abelian. In the basis specified above, the cup-product map $\mu : H^1(G) \wedge H^1(G) \to H^2(G)$ is given by*

$$\mu(e_i \wedge e_j) = \sum_{k=1}^m \epsilon_{i,j}(r_k)\gamma_k.$$

Proof. This follows immediately from [13], Theorem 2.3. Q.E.D.

2.9. Links in $\mathbb{S}^3$

We conclude this section with a classical example. Let L be an oriented link in $\mathbb{S}^3$, with components $L_1, \dots L_n$. Its complement, $X = \mathbb{S}^3 \setminus \bigcup_i L_i$, has the homotopy type of a connected, 2-dimensional finite CW-complex. The homology groups of X are computed by Alexander duality: $H_1(X) = \mathbb{Z}^n$, $H_2(X) = \mathbb{Z}^{n-1}$. It follows that Condition (A) is always satisfied for a link complement. If $L = \hat{\beta}$ is the closure of a pure braid $\beta \in P_n$, then X satisfies the assumption of Proposition 2.7, and so $G = \pi_1(X)$ is a commutator-relators group, with presentation $G = \langle x_1, \dots, x_n \mid \beta(x_i)x_i^{-1} = 1,\ 1 \le i < n\rangle$.

For an arbitrary link L, let $\{e_1, \dots, e_n\}$ be the basis for $H^1(X)$ dual to the meridians of L. Choose arcs $c_{i,j}$ in X connecting L_i to L_j, and let $\gamma_{i,j} \in H^2(X)$ be their duals. Then $\{\gamma_{1,n}, \dots, \gamma_{n-1,n}\}$ forms a basis for $H_2(X)$. Let $l_{i,j} = \operatorname{lk}(L_i, L_j)$ be the linking numbers of L. A presentation for the cohomology ring of X is given by:

$$H^*(X) = \left(e_i, \gamma_{i,j} \;\middle|\; \begin{array}{l} e_ie_j = l_{i,j}\gamma_{i,j},\ \gamma_{i,j} + \gamma_{j,k} + \gamma_{k,i} = 0 \\ \gamma_{i,i} = \gamma_{i,j}e_k = e_k\gamma_{i,j} = \gamma_{i,j}\gamma_{k,l} = 0 \end{array} \right). \tag{7}$$

Let $\mathcal{G}$ be the "linking graph" associated to L: It is the complete graph on n vertices, with edges labelled by the linking numbers. If $\mathcal{G}$ possesses a spanning tree T with n vertices, and all edges labelled ± 1, we say that L is *(cohomologically) rigid*. The complement of such a link satisfies condition (B), see [22, 17, 1]. Moreover, G_2/G_3 is free abelian of rank $\binom{n-1}{2}$, with basis $\{x_{ij} \mid ij \notin T \text{ and } i < j\}$. The classifying map $\chi\colon \bigwedge^2 H \to G_2/G_3$ is given by

$$\chi(e_i \wedge e_j) = \begin{cases} x_{ij} & \text{if } ij \notin T, \\ \sum_{\{k \mid ik \notin T\}} l_{i,k}x_{ik} & \text{if } ij \in T, \end{cases}$$

where $x_{ik} = -x_{ki}$, for $i > k$.

We will be mainly interested in those rigid links for which $l_{i,j} = \pm 1$. Examples include the Hopf links, and, more generally, the singularity links of 2-arrangements in $\mathbb{R}^4$ (see 6.1). For such links, the presentation (7) simplifies to:

$$H^*(X) = \left(e_i \;\middle|\; \begin{array}{l} e_i^2 = 0,\ e_ie_j = -e_je_i, \\ l_{i,j}e_ie_j + l_{j,k}e_je_k + l_{k,i}e_ke_i = 0 \end{array} \right). \tag{8}$$

Moreover, the transpose of the classifying map, $\chi^\top : G_2/G_3 \to \bigwedge^2 H$, is given by the simple formula

$$\chi^\top(x_{ij}) = (e_i - l_{i,j}e_n) \wedge (e_j - l_{i,j}e_n). \tag{9}$$

§3. Resonance varieties

In this section, we define the 'resonance' varieties of the cohomology ring of a space X. We then show that, under certain conditions on X, these varieties are the determinantal varieties of the linearized Alexander matrix of the group $G = \pi_1(X)$.

3.1. Filtration of first cohomology

Let X be a space that satisfies conditions (A_2) and (B_2) of Section 1, and the hypothesis of Proposition 2.7. We thus have: $H^1(X) = \mathbb{Z}^n$, $H^2(X) = \mathbb{Z}^m$, the cup-product map $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$ is surjective, and $G = \pi_1(X)$ is a commutator-relators group.

Lemma 3.2. *Let X be as above, and let $\mathbb{K}$ be a commutative field.*

(a) *The $\mathbb{K}$-cup products may be computed from the integral ones: $\mu_{\mathbb{K}} = \mu \otimes \mathrm{id}_{\mathbb{K}}$.*

(b) *If $H^{\leq 2}(X) \cong H^{\leq 2}(X')$ then $H^{\leq 2}(X, \mathbb{K}) \cong H^{\leq 2}(X', \mathbb{K})$.*

Proof. Let $\kappa : \mathbb{Z} \to \mathbb{K}$ be the homomorphism given by $\kappa(1) = 1$. From the definitions, the coefficient map $\kappa_* : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{K})$, and the map $\mathrm{id} \otimes \kappa : H^*(X) \otimes \mathbb{Z} \to H^*(X) \otimes \mathbb{K}$ commute with cup products. By the Universal Coefficient Theorem (see [2], Theorem 7.4, p. 282), the map $\upsilon : H^*(X) \otimes \mathbb{K} \to H^*(X, \mathbb{K})$, $\upsilon([z] \otimes k) = [z \otimes k]$ is an isomorphism for $* \leq 2$. Since $\upsilon \circ (\mathrm{id} \otimes \kappa) = \kappa_*$, the map υ also commutes with cup products. The conclusions follow. Q.E.D.

Definition 3.3. Let d be an integer, $0 \leq d \leq n$. The dth *resonance variety* of X (with coefficients in $\mathbb{K}$) is the subvariety of $H^1(X, \mathbb{K}) = \mathbb{K}^n$, defined as follows:

$$\mathcal{R}_d(X, \mathbb{K}) = \left\{ \lambda \in H^1(X, \mathbb{K}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(X, \mathbb{K}) \text{ such that} \\ \dim W = d + 1 \text{ and } \mu(\lambda \wedge W) = 0 \end{array} \right\}.$$

The resonance varieties form a descending filtration $\mathbb{K}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_{n-1} \supset \mathcal{R}_n = \emptyset$. The *ambient type* of the $\mathbb{K}$-resonance varieties depends only on the truncated cohomology ring $H^{\leq 2}(X, \mathbb{K})$, and thus, by Lemma 3.2 (b), only on $H^{\leq 2}(X)$. More precisely, if $H^{\leq 2}(X) \cong H^{\leq 2}(X')$, there exists a linear automorphism of $\mathbb{K}^n$ taking $\mathcal{R}_d(X, \mathbb{K})$ to $\mathcal{R}_d(X', \mathbb{K})$.

For a group G, define the resonance varieties to be those of the corresponding Eilenberg-MacLane space: $\mathcal{R}_d(G, \mathbb{K}) := \mathcal{R}_d(K(G, 1), \mathbb{K})$.

Proposition 3.4. *Let X be a space satisfying conditions (A_2) and (B_2). Let $G = \pi_1(X)$. Then $\mathcal{R}_d(X, \mathbb{K}) = \mathcal{R}_d(G, \mathbb{K})$.*

Proof. By Proposition 1.6, the inclusion $j : X \to K(G,1)$ induces an isomorphism $j^* : H^{\leq 2}(G) \to H^{\leq 2}(X)$. The conclusion follows from Lemma 3.2 (b) above. Q.E.D.

3.5. Alexander matrices

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group. Recall the projection map $\phi : \mathbb{F}(n) \to G$, and the abelianization map, $\alpha : G \to \mathbb{Z}^n$, given by $\alpha(x_i) = t_i$.

Definition 3.6. The *Alexander matrix* of G is the $m \times n$ matrix A $=(\alpha\phi\partial_i(r_k))$ with entries in the Laurent polynomials ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Now let $\psi : \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{Z}[[s_1, \dots, s_n]]$ be the ring homomorphism given by $\psi(t_i) = 1 + s_i$ and $\psi(t_i^{-1}) = \sum_{q \geq 0} (-1)^q s_i^q$. Also, let $\psi^{(q)}$ be the graded q^{th} piece of ψ. Since all the relators of G are commutators, the entries of A are in the ideal $(t_1 - 1, \dots, t_n - 1)$, and so $\psi^{(0)} A$ is the zero matrix.

Definition 3.7. The *linearized Alexander matrix* of G is the $m \times n$ matrix

$$M = \psi^{(1)} A.$$

Note that the entries of M are integral linear forms in $s_1, \dots, s_n$. By Lemma 2.4 (a), (b) we have $\psi^{(1)} \alpha\phi\partial_i(r_k) = \psi^{(1)} \alpha\phi\partial_i(r_k^{(3)})$. Thus, M depends only on the relators of G, modulo length 3 commutators. By Lemma 2.4 (c), (d) those truncated relators are given by $r_k^{(3)} = \prod_{i<j} [x_i, x_j]^{\epsilon_{i,j}(r_k)}$. Thus, the entries of M are:

$$M_{k,j} = \sum_{i=1}^{n} \epsilon_{i,j}(r_k) s_i. \tag{10}$$

The linearized Alexander matrix of a link was first considered by Traldi [31]. If the link L has n components, then M has size $n \times (n-1)$, and its entries are $M_{k,j} = l_{k,j} s_k - \delta_{k,j} (\sum_i l_{k,i} s_i)$.

3.8. Equations for resonance varieties

We now find explicit equations for the varieties $\mathcal{R}_d(X, \mathbb{K})$. In view of Proposition 3.4, that is the same as finding equations for $\mathcal{R}_d(G, \mathbb{K})$, with $G = \pi_1(X)$. Moreover, in view of Lemma 3.2 (a), the formula for the integral cup products from Proposition 2.8 may be used to compute the $\mathbb{K}$-cup products. We will use the notations of that proposition for the rest of this section.

Let M be the linearized Alexander matrix of G. Let $M_{\mathbb{K}}$ be the corresponding matrix of linear forms over $\mathbb{K}$, and let $M(\lambda)$ be the matrix $M_{\mathbb{K}}$ evaluated at $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$.

Theorem 3.9. *For G a commutator-relators group with $H_2(G)$ torsion free,*

$$\mathcal{R}_d(G, \mathbb{K}) = \{\lambda \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} M(\lambda) < n - d\}.$$

Proof. Let $\lambda = \sum_{i=1}^n \lambda_i e_i \in H^1(G, \mathbb{K}) = \mathbb{K}^n$. We are looking for $v = \sum_{i=1}^n v_i e_i$ such that $\mu(\lambda \wedge v) = 0$ in $H^2(G, \mathbb{K}) = \mathbb{K}^m$. Recall from Proposition 2.8 that $\mu(e_i \wedge e_j) = \sum_{k=1}^m \epsilon_{i,j}(r_k)\gamma_k$. It follows that

$$\mu(\lambda \wedge v) = \sum_{k=1}^m \Bigg(\sum_{1 \le i,j \le n} \lambda_i v_j \epsilon_{i,j}(r_k) \Bigg) \gamma_k.$$

We thus obtain a linear system of m equations in $v_1, \dots, v_n$:

$$\sum_{j=1}^n \Bigg(\sum_{i=1}^n \lambda_i \epsilon_{i,j}(r_k) \Bigg) v_j = 0,$$

with coefficient matrix $M(\lambda)$.

Now λ belongs to $\mathcal{R}_d(G, \mathbb{K})$ if and only if the space W of solutions of the linear system $M(\lambda) \cdot v = 0$ is at least $(d+1)$-dimensional. That translates into the condition $\operatorname{rank}_{\mathbb{K}} M(\lambda) < n - d$ of the statement, and we are done. Q.E.D.

We will be mainly interested in the coefficient fields $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{Z}_p$, for some prime p. By the above theorem, the $\mathbb{C}$-resonance varieties have *integral* equations. As we shall see in Section 5, although $\mu_{\mathbb{Z}_p} : H^1(X, \mathbb{Z}_p) \wedge H^1(X, \mathbb{Z}_p) \to H^2(X, \mathbb{Z}_p)$ is the reduction $\bmod p$ of $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$, the variety $\mathcal{R}_d(X, \mathbb{Z}_p)$ is *not* necessarily the reduction $\bmod p$ of $\mathcal{R}_d(X, \mathbb{C})$.

Example 3.10. Let X be the complement of an n-component rigid link. The matrix $M(\lambda)$ has entries $M(\lambda)_{k,j} = l_{k,j}\lambda_k - \delta_{k,j}(\sum_i l_{k,i}\lambda_i)$. The variety $\mathcal{R}_1(X, \mathbb{K})$ is the zero-locus of a degree $n-2$ homogeneous polynomial obtained by taking the greatest common divisor of the $(n-1) \times (n-1)$ minors of the matrix $M(\lambda)$. At the other extreme, we have $\mathcal{R}_{n-1}(X, \mathbb{K}) = \{\mathbf{0}\}$. Indeed, the off-diagonal entries of $M(\lambda)$ corresponding to the edges of the maximal spanning tree generate the maximal ideal $(\lambda_1, \dots, \lambda_n)$ of $\mathbb{K}[\lambda_1, \dots, \lambda_n]$.

3.11. Projectivized resonance varieties

The affine variety $\mathcal{R}_d(X,\mathbb{K}) \subset \mathbb{K}^n$ is homogeneous, and so defines a projective variety $\mathcal{P}_d(X,\mathbb{K}) \subset \mathbb{P}(\mathbb{K}^n)$. If $H^{\leq 2}(X)$ is isomorphic to $H^{\leq 2}(X')$, there is a projective automorphism $\mathbb{P}(\mathbb{K}^n) \to \mathbb{P}(\mathbb{K}^n)$ taking $\mathcal{P}_d(X,\mathbb{K})$ to $\mathcal{P}_d(X',\mathbb{K})$. The rest of the above discussion applies to the projective resonance varieties in an obvious manner. In particular, we have:

Corollary 3.12. $\mathcal{P}_d(G,\mathbb{K})=\{\lambda \in \mathbb{P}(\mathbb{K}^n) \,|\, \operatorname{rank}_{\mathbb{K}} M(\lambda) < n-d-1\}$.

§4. Prime index normal subgroups

In this section, we consider nilpotent quotients of commutator-relators groups. We show how to count the normal subgroups of prime index, according to their abelianization.

4.1. Counting subgroups

Let G be a group. For a prime number p, let $\Sigma_p(G)$ be the set of index p normal subgroups of G, and let $N_p(G) = |\Sigma_p(G)|$ be its cardinality.

Proposition 4.2. *For the free group $\mathbb{F}(n)$, the set $\Sigma_p(\mathbb{F}(n))$ is in bijective correspondence with the projective space $\mathbb{P}(\mathbb{Z}_p^n)$.*

Proof. Every index p normal subgroup of $\mathbb{F}_n$ is the kernel of an epimorphism $\lambda : \mathbb{F}(n) \to \mathbb{Z}_p$. Such homomorphisms are parametrized by $\mathbb{Z}_p^n \setminus \{\mathbf{0}\}$. Two epimorphisms λ and λ' have the same kernel if and only if $\lambda = q \cdot \lambda'$, for some $q \in \mathbb{Z}_p^*$. Q.E.D.

Corollary 4.3. *Let $G = \mathbb{F}(n)/R$ be a commutator-relators group. For all primes p,*

$$N_p(G) = \frac{p^n - 1}{p - 1}.$$

Proof. Since R consists of commutators,

$$\operatorname{Hom}(G,\mathbb{Z}_p) \cong \operatorname{Hom}(\mathbb{F}(n),\mathbb{Z}_p).$$

Thus, $\Sigma_p(G)$ is in one-to-one correspondence with $\Sigma_p(\mathbb{F}(n)) = \mathbb{P}(\mathbb{Z}_p^n)$. Q.E.D.

4.4. Abelianizing normal subgroups

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group. Let $K \lhd G$ be a normal subgroup of index p, defined by a representation $\lambda : G \to \mathbb{Z}_p$, $\lambda(x_i) = \lambda_i$. Let $\bar{\lambda} : \mathbb{Z}G \to \mathbb{Z}\mathbb{Z}_p$ be the linear extension of λ to group rings. More precisely, we view here $\mathbb{Z}_p$ as a multiplicative group, with generator ζ. Then $\bar{\lambda}(x_i) = \zeta^{\lambda_i}$. Finally, let $\beta : \mathbb{Z}\mathbb{Z}_p \to \mathrm{Mat}(p, \mathbb{Z})$ be the ring homomorphism defined by the (left) regular representation of $\mathbb{Z}_p$.

Definition 4.5. For a given representation $\lambda : G \to \mathbb{Z}_p$, the *twisted Alexander matrix of* G is the $pm \times pn$ matrix

$$A_\lambda = \big(\bar{\lambda}\phi\partial_i(r_k)\big)^\beta$$

obtained from $\big(\bar{\lambda}\phi\partial_i(r_k)\big)$ by replacing each entry e with $\beta(e)$.

Proposition 4.6. *Let G be a commutator-relators group, and let $K = \ker(\lambda : G \to \mathbb{Z}_p)$. The matrix A_λ is a relation matrix for the group $H_1(K) \oplus \mathbb{Z}^{p-1}$.*

A proof can be found in [16]. The matrix A_λ is equivalent (via row-and-column operations) to a diagonal matrix, from which the rank and elementary divisors of $H_1(K)$ can be read off.

4.7. Nilpotent quotients

We now apply the above procedure to a particular class of groups: the nilpotent quotients G/G_q, $q \geq 3$, of a commutator-relators group $G = \mathbb{F}(n)/R$.

Let $\lambda : G/G_q \to \mathbb{Z}_p$ be a non-trivial representation. To describe explicitly the presentation matrix A_λ of Proposition 4.6, we need to examine more closely the Fox derivatives of the relators c_h and $r_k^{(q)}$ in the presentation (6) for G/G_q.

If c is a non-simple basic commutator, then Lemma 2.4 (a), (b) gives $\bar{\lambda}\phi(\partial_i c) = 0$. If $c = [x_{i_1}, x_{i_2}, \dots, x_{i_q}]$ is a simple commutator, then it follows from Lemma 2.4 (b) that $\bar{\lambda}\phi\,(\partial_i c)$ is either zero or of the form $e = \pm(\zeta^{a_1} - 1) \cdots (\zeta^{a_{q-2}} - 1) \in \mathbb{Z}\mathbb{Z}_p$, for some integers $1 \leq a_j \leq p - 1$.

Recall that the truncation $r_k^{(q)}$ is a product of basic commutators of length $< q$. The same argument shows that $\bar{\lambda}\phi\big(\partial_i r_k^{(q)}\big)$ is a linear combination of elements in $\mathbb{Z}\mathbb{Z}_p$ of the form $(\zeta^{a_{i_1}} - 1) \cdots (\zeta^{a_{i_j}} - 1)$, for $j < q - 2$.

The following lemma shows the typical simplifications that we will perform on $\big(\bar{\lambda}\phi(\partial c_h)\big)^\beta$ and $\big(\bar{\lambda}\phi(\partial r_k^{(q)})\big)^\beta$.

Lemma 4.8. *The integral $p \times p$ matrix e^β corresponding to $e = (\zeta^{a_1} - 1) \cdots (\zeta^{a_k} - 1) \in \mathbb{Z}\mathbb{Z}_p$ has diagonal form*

$$\big(\underbrace{p^{r-1}, \dots, p^{r-1}}_{p-l-1}, \underbrace{p^r, \dots, p^r}_{l}, 0\big),$$

where $r = \lceil \frac{k-1}{p-1} \rceil$, and $l = k - 1 - (r-1)(p-1)$. Moreover, there is a sequence of row and column operations, independent of the particular e, that brings e^β to that diagonal form.

Proposition 4.9. *Let K be an index p normal subgroup of the free nilpotent quotient $\mathbb{F}(n)/\mathbb{F}(n)_q$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus (\mathbb{Z}/p^{r-1}\mathbb{Z})^{(n-1)(p-l-1)} \oplus (\mathbb{Z}/p^r\mathbb{Z})^{(n-1)l},$$

where $r = \left\lceil \frac{q-2}{p-1} \right\rceil$, and $l = q - 2 - (r-1)(p-1)$.

Proof. In this case, only commutator relators are present, so Lemma 4.8, applied to each entry $\bar{\lambda}\phi(\partial c_h)$, shows that the matrix A_λ is equivalent the following diagonal matrix:

$$D = \big(\underbrace{p^{r-1}, \dots, p^{r-1}}_{(n-1)(p-l-1)}, \underbrace{p^r, \dots, p^r}_{(n-1)l}, \underbrace{0, \dots, 0}_{n+p-1}\big). \tag{11}$$

Q.E.D.

Theorem 4.10. *Let $G = \mathbb{F}(n)/R$ be a commutator-relators group. Let K be an index p normal subgroup of G/G_q. Set $r = \left\lceil \frac{q-2}{p-1} \right\rceil$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus \bigoplus_{i=0}^{r} (\mathbb{Z}/p^i\mathbb{Z})^{d_i},$$

for some positive integers $d_0, \dots, d_r$ such that $d_0 + \cdots + d_r = (n-1)(p-1)$ and $d_r \le l(n-1)$.

Proof. Let $K = \ker(\lambda : G/G_q \to \mathbb{Z}_p)$. Consider the relation matrix A_λ, corresponding to the presentation $G/G_q = \mathbb{F}/R\mathbb{F}_q$ from (6). Partition A_λ into two blocks, $A_\lambda = \left(\begin{smallmatrix} B_\lambda \\ C_\lambda \end{smallmatrix}\right)$, where B_λ corresponds to the relators R, and C_λ corresponds to the basic commutators.

Assume that the row and column operations of Lemma 4.8 have already been performed. Then, after moving all the zero columns to the right, A_λ is equivalent to $\left(\begin{smallmatrix} B'_\lambda & 0 \\ D' & 0 \end{smallmatrix}\right)$, where $D = (D' \ 0)$ is the diagonal matrix (11). Since the number of zero diagonal elements of D is $n+p-1$, the rank of K is n. Since the non-zero diagonal elements of D are either p^{r-1} or p^r, the elementary divisors of K are among $p, p^2, \dots, p^r$. The conclusion readily follows. Q.E.D.

4.11. ν-Invariants

In view of Theorem 4.10, we define the following numerical invariants of isomorphism type for the nilpotent quotients of a group.

Definition 4.12. Let G be a commutator-relators group, and let G/G_q be the $(q-1)^{\text{st}}$ nilpotent quotient of G. Given a prime p, and a positive integer d, define

$$\nu_{p,d}(G/G_q) = \#\left\{ K \lhd G/G_q \;\middle|\; \begin{array}{l} [G/G_q : K] = p \text{ and} \\ \dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{Z}_p = d \end{array} \right\}.$$

Example 4.13. If $q = 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \leq d \leq n-1$. So we have invariants $\nu_{p,0}(G/G_3), \ldots, \nu_{p,n-1}(G/G_3)$ for the second nilpotent quotient of G. Since $\sum_{d=0}^{n-1} \nu_{p,d} = \frac{p^n-1}{p-1}$, it is enough to compute $\nu_{p,1}, \ldots, \nu_{p,n-1}$.

Example 4.14. If $q = 4$, and $p \geq 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \leq d \leq 2n-2$. If $p = 2$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_2^{d_1} \oplus \mathbb{Z}_4^{d_2}$, for some $0 \leq d = d_1 + d_2 \leq n-1$.

4.15. Second nilpotent quotient

We now restrict our attention to G/G_3. From (6), for $q = 3$ we obtain the presentation:

$$G/G_3 = \langle x_1, \ldots, x_n \mid r_1^{(3)}, \ldots, r_m^{(3)}, c_1, \ldots, c_l \rangle, \tag{12}$$

where $l = 2\binom{n+1}{3}$, and $c_1, \ldots, c_l$ are the basic commutators $[x_i, [x_j, x_k]]$, with $j < k$ and $i \geq j$.

Theorem 4.16. *Given an epimorphism $\lambda : G/G_3 \to \mathbb{Z}_p$, with kernel K_λ, we have*

$$\dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K_\lambda)) \otimes \mathbb{Z}_p = n - 1 - \operatorname{rank}_{\mathbb{Z}_p} M(\lambda).$$

Proof. Recall from the proof of Theorem 4.10 that the relation matrix of the abelian group $H_1(K_\lambda)$ has the following form: $A_\lambda = \begin{pmatrix} B'_\lambda & 0 \\ C'_\lambda & 0 \end{pmatrix}$. We have already seen in Proposition 4.9 that C'_λ is equivalent to a diagonal matrix $D' = \begin{pmatrix} I_{(n-1)(p-2)} & 0 \\ 0 & p \cdot I_{n-1} \end{pmatrix}$.

Recall also that $r_k^{(3)} = \prod_{i<j} [x_i, x_j]^{\epsilon_{i,j}(r_k)}$. A computation using formula (a) in Lemma 2.4 shows:

$$\bar{\lambda}\phi\left(\frac{\partial r_k^{(3)}}{\partial x_l}\right) = \sum_{i=1}^{n} \epsilon_{i,l}(r_k)(\zeta^{\lambda_i} - 1), \tag{13}$$

for $1 \leq l \leq n$ and $1 \leq k \leq m$.

Consider $e = \sum_{\sigma=1}^{p-1} a_\sigma(\zeta^\sigma - 1) \in \mathbb{Z}\mathbb{Z}_p$. Set $a = \sum_{\sigma=1}^{p-1} a_\sigma$. It is readily seen that the matrix e^β is equivalent to:

$$\begin{pmatrix} * & \cdots & * & p \cdot a & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ * & \cdots & * & p \cdot a & 0 \\ * & \cdots & * & \sum_{\sigma=1}^{p-1} a_\sigma \cdot \sigma & 0 \end{pmatrix}. \tag{14}$$

Now (10), together with (13) and (14), imply that B'_λ is equivalent to $\left(\begin{smallmatrix} * & 0 \\ * & M(\lambda)' \end{smallmatrix}\right)$, where $M(\lambda)'$ is some codimension 1 minor of $M(\lambda)$. Hence, A_λ is equivalent to:

$$\begin{pmatrix} * & 0 & 0 \\ * & M(\lambda)' & 0 \\ I_{(n-1)(p-2)} & 0 & 0 \\ 0 & p \cdot I_{n-1} & 0 \end{pmatrix}.$$

The theorem now follows from the following fact: An integral matrix of the form $\left(\begin{smallmatrix} Q \\ p \cdot I_n \end{smallmatrix}\right)$ is equivalent to $\left(\begin{smallmatrix} I_r & 0 \\ 0 & p \cdot I_d \end{smallmatrix}\right)$, where $r = \operatorname{rank}_{\mathbb{Z}_p} Q \otimes \mathrm{id}_{\mathbb{Z}_p}$, and $d = n - r$. Q.E.D.

Corollary 4.17. $\nu_{p,d}(G/G_3) = \#\{K_\lambda \in \Sigma_p(G/G_3) \mid \operatorname{rank}_{\mathbb{Z}_p} M(\lambda) = n - d - 1\}$.

4.18. Resonance varieties and subgroups of G/G_3

The following theorem relates the distribution of index p normal subgroups of G/G_3, according to their abelianization, to the number of points on the n-dimensional projective space over $\mathbb{Z}_p$, according to the stratification by the resonance varieties.

Theorem 4.19. *For G a commutator-relators group with $H_2(G)$ torsion free,*

$$\nu_{p,d}(G/G_3) = \#(\mathcal{P}_d(G, \mathbb{Z}_p) \setminus \mathcal{P}_{d+1}(G, \mathbb{Z}_p)).$$

Proof. Follows from Corollary 3.12 and Corollary 4.17. Q.E.D.

§5. Complex arrangements

We illustrate the techniques developed in the previous sections with the main example of spaces satisfying conditions (A) and (B): complements of complex hyperplane arrangements.

5.1. Cohomology and fundamental group

Let $\mathcal{A}'$ be a complex hyperplane arrangement, with complement X'. Let $\mathcal{A}$ be a generic two-dimensional section of $\mathcal{A}'$, with complement X. Then, by the Lefschetz-type theorem of Hamm and Lê [15], the inclusion $i : X \to X'$ induces an isomorphism $i_* : \pi_1(X) \to \pi_1(X')$ and a monomorphism $i^* : H^2(X') \to H^2(X)$. By the Brieskorn-Orlik-Solomon theorem, the map i^* is, in fact, an isomorphism. So, for our purposes here, we may restrict our attention to $\mathcal{A}$.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n affine lines in $\mathbb{C}^2$, in general position at infinity. Let $v_1, \dots, v_s$ be the intersection points of the lines. If $v_q = H_{i_1} \cap \cdots \cap H_{i_m}$, set $V_q = \{i_1, \dots, i_m\}$ and $\overline{V}_q = V_q \setminus \{\max V_q\}$. The level 2 of the lattice of $\mathcal{A}$ is encoded in the list $L_2(\mathcal{A}) = \{V_1, \dots, V_s\}$, which keeps track of the incidence relations between the points and the lines of the arrangement.

The following properties hold:

(i) The homology groups of $X = \mathbb{C}^2 \setminus \bigcup_i H_i$ are free abelian, of ranks $b_1 = n$, $b_2 = \sum_{q=1}^{s} \left|\overline{V}_q\right|$, and $b_i = 0$ for $i > 2$. The cohomology ring is determined by $L_2(\mathcal{A})$ (see [26]):

$$H^*(X) = \left(e_1, \dots, e_n \;\middle|\; \begin{array}{l} e_i^2 = 0,\ e_i e_j = -e_j e_i \\ e_i e_j + e_j e_k + e_k e_i = 0 \ \text{ for } i, j, k \in V_q,\ 1 \le q \le s \end{array} \right).$$

(ii) The fundamental group $G = \pi_1(X)$ is a commutator-relators group:

$$G = \langle x_1, \dots, x_n \mid \beta_q(x_i) x_i^{-1} = 1 \quad \text{for } i \in \overline{V}_q \text{ and } q = 1, \dots, s \rangle.$$

The pure braid monodromy generators $\beta_1, \dots, \beta_s$ can be read off from a 'braided wiring diagram' associated to $\mathcal{A}$ (see [5]). Moreover, the space X is homotopy equivalent to the 2-complex given by this presentation (see [18]).

(iii) The second nilpotent quotient is determined by $L_2(\mathcal{A})$:

$$G/G_3 = \left\langle x_1, \dots, x_n \;\middle|\; \begin{array}{ll} [x_i, \prod_{j \in V_q} x_j] & \text{for } i \in \overline{V}_q, \quad 1 \le q \le s \\ [x_i, [x_j, x_k]] & \text{for } 1 \le j < k \le n, \quad j \le i \le n \end{array} \right\rangle.$$

This follows from the presentation in (ii), together with (12) (see also [28]).

(iv) The linearized Alexander matrix is determined by $L_2(\mathcal{A})$. It is obtained by stacking $M_{V_1}(\lambda), \dots, M_{V_s}(\lambda)$, where $M_V(\lambda)$ is the

$\left|\overline{V}\right| \times n$ matrix with entries

$$M_V(\lambda)_{i,j} = \delta_{j,V}\Big(\lambda_i - \delta_{i,j}\sum_{k\in V}\lambda_k\Big), \qquad \text{for } i \in \overline{V} \text{ and } 1 \le j \le n.$$

For a detailed discussion of the Alexander matrix and the Alexander invariant of $\mathcal{A}$, see [6].

From properties (i) and (ii), we deduce that X satisfies the conditions from Proposition 2.7.

5.2. Resonance varieties over $\mathbb{C}$

The resonance varieties of a complex hyperplane arrangement were introduced by Falk in [11]. Let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$, in general position at infinity. Set $\mathcal{R}_d(\mathcal{A}) := \mathcal{R}_d(X, \mathbb{C})$. By Theorem 3.1 in [11], this definition agrees with Falk's definition.

Qualitative results as to the nature of the resonance varieties of complex arrangements were obtained by a number of authors, [33, 11, 7, 19, 20]. We summarize some of those results, as follows.

Theorem 5.3. *Let $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^n$ be the resonance variety of an arrangement of n complex hyperplanes. Then*:

(a) *The ambient type of $\mathcal{R}_1(\mathcal{A})$ determines the isomorphism type of $H^{\le 2}(X)$.*
(b) *$\mathcal{R}_1(\mathcal{A})$ is contained in the hyperplane $\Delta_n := \{\sum_{i=1}^n \lambda_i = 0\}$.*
(c) *Each component C_i of $\mathcal{R}_1(\mathcal{A})$ is a linear subspace.*
(d) *$C_i \cap C_j = \{\mathbf{0}\}$ for $i \ne j$.*
(e) *$\mathcal{R}_d(\mathcal{A}) = \{\mathbf{0}\} \cup \bigcup_{\dim C_i \ge d+1} C_i$.*

Proof. Part (a) was proved in [11]. Part (b) was proved in [33] and [11]. Part (c) was conjectured in [11], and proved in [7] and [19]. Part (d) is proved in [20]. Part (e) follows from [20], Theorem 3.4, as was pointed out to us by S. Yuzvinsky. Q.E.D.

By Theorem 3.9, the resonance varieties $\mathcal{R}_d(\mathcal{A})$ are the determinantal varieties associated to the linearized Alexander matrix, M. For another set of explicit equations, obtained from a presentation of the linearized Alexander invariant, see [7].

All the components of $\mathcal{R}_1(\mathcal{A})$ arise from *neighborly partitions* of subarrangements of $\mathcal{A}$, see [11], [20]. To a partition Π of $\mathcal{A}' \subset \mathcal{A}$, such that a certain bilinear form associated to Π is degenerate, there corresponds a component C_Π of $\mathcal{R}_1(\mathcal{A})$. For each $V \in L_2(\mathcal{A})$ with $|V| \ge 3$, there is a *local* component, $C_V = \Delta_n \cap \{\lambda_i = 0 \mid i \notin V\}$, of dimension $|V| - 1$, corresponding to the partition (V) of $\mathcal{A}_V = \{H_i \mid i \in V\}$. The

other components of $\mathcal{R}_1(\mathcal{A})$ are called *non-local*. For more details and examples, see [11, 7, 19, 20].

5.4. Resonance varieties over $\mathbb{Z}_p$

We now turn to the characteristic p resonance varieties, $\mathcal{R}_d(\mathcal{A};\mathbb{Z}_p)$. Recall that the variety $\mathcal{R}_d(\mathcal{A})$ has integral equations, so we may consider its reduction mod p. As we shall see, there are arrangements $\mathcal{A}$ such that $\mathcal{R}_d(\mathcal{A};\mathbb{Z}_p)$ does not coincide with $\mathcal{R}_d(\mathcal{A})$ mod p, for certain primes p. Indeed:

- The number of irreducible components, or the dimensions of the components may be different, as illustrated in Examples 5.9 and 5.10 below.
- The analogues of Theorem 5.3 (a) and (e) fail in general, as seen in Examples 5.8 and 5.10 below.

On the other hand, it seems likely that the analogues of Theorem 5.3 (b), (c) and (d) hold for every prime p.

Now let $\nu_{p,d}(\mathcal{A}) = \nu_{p,d}(G/G_3)$ be the number of normal subgroups of G/G_3 with abelianization $\mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for $0 \le d \le n-1$. By properties (i) and (ii) above, Theorem 4.19 applies, and so $\nu_{p,d}(\mathcal{A})$ can be computed from the $\mathbb{Z}_p$-resonance varieties.

Corollary 5.5. *If $\mathcal{R}_d(\mathcal{A},\mathbb{Z}_p) = \mathcal{R}_d(\mathcal{A})$ mod p, for all $d \ge 1$, then*

$$\nu_{p,d-1}(\mathcal{A}) = \frac{p^d-1}{p-1} m_d,$$

where m_d is the number of components of $\mathcal{R}_1(\mathcal{A})$ of dimension d.

Proof. From the assumption, properties (c)–(e) hold for $\mathcal{R}_d(\mathcal{A},\mathbb{Z}_p)$. Therefore, $\mathcal{P}_d(X,\mathbb{Z}_p)\backslash\mathcal{P}_{d+1}(X,\mathbb{Z}_p)$ consists of m_d disjoint, d-dimensional projective subspaces in $\mathbb{P}(\mathbb{Z}_p^n)$. The formula follows from Theorem 4.19. Q.E.D.

If all the components of $\mathcal{R}_1(\mathcal{A})$ are local, then $m_d = \#\{V \in L_2(\mathcal{A}) \mid |V| = d+1\}$, but the Corollary may not apply, see Example 5.9.

5.6. Examples

We conclude this section with a few examples that illustrate the phenomena mentioned above. The motivation to look at Examples 5.8 and 5.10 came from S. Yuzvinsky, who was the first to realize that there are exceptional primes for these arrangements. His method of computing the corresponding non-local components is different from ours, though.

Example 5.7. Let $\mathcal{A}$ be the reflection arrangement of type A_3, with lattice

$$L_2(\mathcal{A}) = \{123, 145, 246, 356, 16, 25, 34\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 5 components of dimension 2. The non-local component, $C_\Pi = \{\lambda_1 - \lambda_6 = \lambda_2 - \lambda_5 = \lambda_3 - \lambda_4 = 0\} \cap \Delta_6$, corresponds to the partition $\Pi = (16 \mid 25 \mid 34)$, see [11, 7, 19].

For all primes p, Corollary 5.5 applies, giving $\nu_{p,1} = 5(p+1)$.

Example 5.8. Let $\mathcal{A}$ be the realization of the non-Fano plane, with lattice

$$L_2(\mathcal{A}) = \{123, 147, 156, 257, 345, 367, 24, 26, 46\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 9 components of dimension 2. The non-local components are given by the partitions $\Pi_1 = (13 \mid 46 \mid 57)$, $\Pi_2 = (15 \mid 24 \mid 37)$, $\Pi_3 = (17 \mid 26 \mid 35)$ of the corresponding type A_3 sub-arrangements, see [7].

For $p > 2$, Corollary 5.5 applies, and so $\nu_{p,1} = 9(p+1)$.

For $p = 2$, though, $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_2)$ has a single, 3-dimensional non-local component, $C_\Pi = \{\lambda_1 + \lambda_4 + \lambda_7 = \lambda_2 + \lambda_5 + \lambda_7 = \lambda_3 + \lambda_6 + \lambda_7 = 0\} \cap \Delta_7$, corresponding to $\Pi = (1 \mid 3 \mid 5 \mid 7 \mid 246)$. Furthermore, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_2)$ has a single, 1-dimensional component, $C_{\Pi'} = \{\lambda_1 + \lambda_7 = \lambda_3 + \lambda_7 = \lambda_5 + \lambda_7 = \lambda_2 = \lambda_4 = \lambda_6 = 0\}$, corresponding to $\Pi' = (1 \mid 3 \mid 5 \mid 7)$, and $\mathcal{R}_3(\mathcal{A}, \mathbb{Z}_2) = \{\mathbf{0}\}$. Thus, $\nu_{2,1} = 24$ and $\nu_{2,2} = 1$.

Example 5.9. Let $\mathcal{A}$ be one of the realizations of the MacLane matroid, with

$$L_2(\mathcal{A}) = \{123, 456, 147, 267, 258, 348, 357, 168, 15, 24, 36, 78\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 8 local components. Despite the fact that $\mathcal{A}$ supports many neighborly partitions, $\mathcal{R}_1(\mathcal{A})$ has no non-local components, since Falk's degeneracy condition is not satisfied, see [11].

For $p \neq 3$, Corollary 5.5 applies, and so $\nu_{p,1} = 8(p+1)$.

For $p = 3$, though, the degeneracy condition is satisfied, and the variety $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_3)$ has a non-local, 2-dimensional component,

$$\begin{aligned} C_\Pi = \{&\lambda_2 + \lambda_5 + \lambda_8 = \lambda_3 + \lambda_5 - \lambda_8 = \lambda_4 - \lambda_5 - \lambda_8 \\ &= \lambda_5 - \lambda_6 - \lambda_8 = \lambda_1 + \lambda_5 = \lambda_7 + \lambda_8 = 0\}, \end{aligned}$$

corresponding to $\Pi = (15 \mid 24 \mid 36 \mid 78)$. Moreover, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_3) = \{\mathbf{0}\}$. Hence, $\nu_{3,1} = 36$.

Example 5.10. Let $\mathcal{A}$ be the realization of the affine plane over $\mathbb{Z}_3$, with lattice

$$L_2(\mathcal{A}) = \{123, 456, 789, 147, 258, 369, 159, 357, 168, 249, 267, 348\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 12 local components, and 4 non-local components of dimension 2, see [11, 7, 19, 20].

For $p \neq 3$, Corollary 5.5 applies, and so $\nu_{p,1} = 16(p+1)$.

On the other hand, $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_3)$ has a single, 3-dimensional non-local component, $C_\Pi = \{\lambda_1 + \lambda_6 + \lambda_8 = \lambda_2 + \lambda_4 + \lambda_9 = \lambda_3 + \lambda_5 + \lambda_7 = \lambda_3 + \lambda_4 + \lambda_8 = \lambda_3 + \lambda_6 + \lambda_9 = \lambda_7 + \lambda_8 + \lambda_9 = 0\}$, corresponding to $\Pi = (123 \mid 456 \mid 789)$, or any other of the partitions that give rise to the 4 non-local components of $\mathcal{R}_1(\mathcal{A})$. Moreover, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_3) = C_\Pi$, and $\mathcal{R}_3(\mathcal{A}, \mathbb{Z}_3) = \{\mathbf{0}\}$. Thus, $\nu_{3,1} = 48$ and $\nu_{3,2} = 13$.

Example 5.11. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be generic plane sections of the two arrangements from [11], Example 4.10. Each arrangement consists of 7 affine lines in $\mathbb{C}^2$, and each resonance variety has only local components. Thus, the ν-invariants of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. On the other hand, as shown by Falk, there is no linear automorphism $\mathbb{C}^7 \to \mathbb{C}^7$ restricting to an isomorphism $\mathcal{R}_1(\mathcal{A}_1) \to \mathcal{R}_1(\mathcal{A}_2)$. The same 'polymatroid' argument shows that there is no automorphism $\mathbb{P}(\mathbb{Z}_p^7) \to \mathbb{P}(\mathbb{Z}_p^7)$ restricting to $\mathcal{P}_1(\mathcal{A}_1, \mathbb{Z}_p) \to \mathcal{P}_1(\mathcal{A}_2, \mathbb{Z}_p)$. Thus, the ambient type of the (projective) resonance varieties carries more information than the count of their points.

§6. Real arrangements

We conclude with an application to the classification of arrangements of transverse planes in $\mathbb{R}^4$. Though similar in some respects to central line arrangements in $\mathbb{C}^2$, such arrangements lack a complex structure. That difference manifests itself in the nature of the resonance varieties.

6.1. Arrangements of real planes

A 2-*arrangement* in $\mathbb{R}^4$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of transverse planes through the origin of $\mathbb{R}^4$. Such an arrangement $\mathcal{A}$ is a realization of the uniform matroid $U_{2,n}$; thus, its intersection lattice is solely determined by n. Let $X = \mathbb{R}^4 \setminus \bigcup_i H_i$ be the complement of the arrangement. The *link* of the arrangement is $L = \mathbb{S}^3 \cap \bigcup_i H_i$. Clearly, the complement of $\mathcal{A}$ deform-retracts onto the complement of L. The link L is the closure of a pure braid in P_n, see [24], [23]. Hence, $G = \pi_1(X)$ is a commutator-relators group.

The *linking numbers* of $\mathcal{A}$ are by definition those of the link L. They can be computed from the defining equations of $\mathcal{A}$: If $H_i = \{\alpha_i = \alpha_i' = 0\}$, for some linear forms $\alpha_i, \alpha_i' : \mathbb{R}^4 \to \mathbb{R}$, then $l_{i,j} = \operatorname{sgn}(\det(\alpha_i, \alpha_i', \alpha_j, \alpha_j'))$, see [34]. A presentation for the cohomology ring of X in terms of the linking numbers is given in (8), see also [34].

Arrangements of transverse planes in $\mathbb{R}^4$ fall, in the terminology of [8], into several types: horizontal and non-horizontal, decomposable and indecomposable. A 2-arrangement $\mathcal{A}$ is *horizontal* if it admits a defining polynomial of the form $f(z,w) = \prod_{i=1}^n (z + a_i w + b_i \bar{w})$, with a_i, b_i real. From the coefficients of f, one reads off a permutation $\tau \in S_n$. Conversely, given τ, there is a horizontal arrangement, $\mathcal{A}(\tau)$, whose associated permutation is τ. A 2-arrangement is *decomposable* if its link is the $(1, \pm 1)$-cable of the link of another 2-arrangement, and it is *completely decomposable* if its link can be obtained from the unknot by successive $(1, \pm 1)$-cablings. See [23] for details.

6.2. Resonance varieties

Let $\mathcal{R}_d(\mathcal{A}) := \mathcal{R}_d(X, \mathbb{C})$ be the dth resonance variety of $\mathcal{A}$. Recall that the resonance varieties form a tower $\mathbb{C}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_{n-1} = \{\mathbf{0}\}$. Moreover, they are the determinantal varieties of the $n \times (n-1)$ matrix $M(\lambda)$, whose entries are given by $M(\lambda)_{k,j} = l_{k,j}\lambda_k - \delta_{k,j}(\sum_i l_{k,i}\lambda_i)$.

If $\mathcal{A}$ is decomposable, the top resonance variety, $\mathcal{R}_1(\mathcal{A})$, contains as a component the hyperplane $\Delta_n = \{\lambda_1 + \cdots + \lambda_n = 0\}$. Moreover, if $\mathcal{A}$ is completely decomposable, $\mathcal{R}_1(\mathcal{A})$ is the union of a central arrangement of $n-2$ hyperplanes in $\mathbb{C}^n$ (counting multiplicities), with defining equations of the form $\epsilon_1\lambda_1 + \cdots + \epsilon_n\lambda_n = 0$, where $\epsilon_i = \pm 1$. If $\mathcal{A}$ is indecomposable, though, $\mathcal{R}_1(\mathcal{A})$ may contain non-linear components (see Example 6.5).

At the other extreme, all the components of the variety $\mathcal{R}_{n-2}(\mathcal{A})$ are linear. It can be shown that a horizontal arrangement $\mathcal{A}$ is indecomposable if and only if $\mathcal{R}_{n-2}(\mathcal{A}) = \{\mathbf{0}\}$.

Example 6.3. In [34], Ziegler provided the first examples of 2-arrangements with isomorphic intersection lattices, but non-isomorphic cohomology rings. Those arrangements are: $\mathcal{A} = \mathcal{A}(1234)$ and $\mathcal{A}' = \mathcal{A}(2134)$. We can distinguish their cohomology rings by counting the components of their resonance varieties:

$$\mathcal{R}_1(\mathcal{A}) = \Delta_4\,, \quad \mathcal{R}_1(\mathcal{A}') = \Delta_4 \cup \{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0\}.$$

The example $\mathcal{A}'$ shows that the analogues of Theorem 5.3 (b), (d), (e) do not hold for 2-arrangements:

- The second component of $\mathcal{R}_1(\mathcal{A}')$ does *not* lie in the hyperplane Δ_4.
- The two components of $\mathcal{R}_1(\mathcal{A}')$ do *not* intersect only at the origin, but rather, in the 2-dimensional subspace $\{\lambda_1+\lambda_2 = \lambda_3+\lambda_4 = 0\}$.
- We have $\mathcal{R}_2(\mathcal{A}') = \{\lambda_1 + \lambda_2 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_1 = \lambda_2 = \lambda_3 + \lambda_4 = 0\}$, and thus the stratification of $\mathcal{R}_1$ by $\mathcal{R}_d$'s is *not* by dimension of components.

Example 6.4. Let $\mathcal{A} = \mathcal{A}(321456)$ and $\mathcal{A}' = \mathcal{A}(213456)$. Then:

$$\begin{aligned}
&\mathcal{R}_1(\mathcal{A}) = \mathcal{R}_1(\mathcal{A}') = \Delta_6 \cup \{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 = 0\},\\
&\mathcal{R}_2(\mathcal{A}) = \mathcal{R}_1(\mathcal{A}), \qquad \mathcal{R}_2(\mathcal{A}') = \Delta_6.
\end{aligned}$$

This example shows that the analogue of Theorem 5.3 (a) does not hold for 2-arrangements: The variety $\mathcal{R}_1$ fails to determine $\mathcal{R}_2$, and thereby fails to determine the cohomology ring of the complement.

Example 6.5. The horizontal arrangement $\mathcal{A}(31425)$ is indecomposable. Its resonance varieties are:

$$\begin{aligned}
\mathcal{R}_1 = \{&\lambda_1^3 - \lambda_2^3 - \lambda_3^3 + \lambda_4^3 - \lambda_5^3 + \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 + \lambda_1^2\lambda_3 - \lambda_1\lambda_3^2 - \lambda_1^2\lambda_4 -\\
&\lambda_1\lambda_4^2 + \lambda_1^2\lambda_5 - \lambda_1\lambda_5^2 + \lambda_2^2\lambda_3 + \lambda_2\lambda_3^2 - \lambda_2^2\lambda_4 + \lambda_2\lambda_4^2 + \lambda_2^2\lambda_5 + \lambda_2\lambda_5^2 -\\
&\lambda_3^2\lambda_4 + \lambda_3\lambda_4^2 + \lambda_3^2\lambda_5 + \lambda_3\lambda_5^2 + \lambda_4^2\lambda_5 - \lambda_4\lambda_5^2 + 2\lambda_1\lambda_2\lambda_3 - 2\lambda_1\lambda_2\lambda_4 +\\
&2\lambda_1\lambda_2\lambda_5 - 2\lambda_1\lambda_3\lambda_4 + 2\lambda_1\lambda_3\lambda_5 - 2\lambda_1\lambda_4\lambda_5 + 2\lambda_2\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_5 +\\
&2\lambda_2\lambda_4\lambda_5 + 2\lambda_3\lambda_4\lambda_5 = 0\}\\
\mathcal{R}_2 = \{&\lambda_1 + \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0\} \cup \{\lambda_1 + \lambda_3 = \lambda_2 = \lambda_4 = \lambda_5 = 0\} \cup\\
&\{\lambda_2 + \lambda_4 = \lambda_1 = \lambda_3 = \lambda_5 = 0\} \cup \{\lambda_3 + \lambda_4 = \lambda_1 = \lambda_2 = \lambda_5 = 0\} \cup\\
&\{\lambda_1 + \lambda_5 = \lambda_2 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_4 + \lambda_5 = \lambda_1 = \lambda_2 = \lambda_3 = 0\} \cup\\
&\{\lambda_1 - \lambda_4 = \lambda_2 = \lambda_3 = \lambda_5 = 0\} \cup \{\lambda_2 - \lambda_3 = \lambda_1 = \lambda_4 = \lambda_5 = 0\} \cup\\
&\{\lambda_2 - \lambda_5 = \lambda_1 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_3 - \lambda_5 = \lambda_1 = \lambda_2 = \lambda_4 = 0\}\\
\mathcal{R}_3 = \{&\mathbf{0}\}
\end{aligned}$$

This example shows that the analogue of Theorem 5.3 (c) does not hold for 2-arrangements: The variety $\mathcal{R}_1$ is not linear.

6.6. Ziegler invariant

The cohomology rings of the arrangements in Example 6.3 were distinguished by Ziegler by means of an invariant closely related to one of the Z-invariants introduced in 1.16.

Recall the sequence $0 \to G_2/G_3 \to G/G_3 \to H \to 0$. This central extension is determined by the map $\chi^\top : G_2/G_3 \to \bigwedge^2 H$, given explicitly

by (9). The invariant $Z_{0,1}(\chi) = \operatorname{coker} \chi^\top$ equals $H^2(G) = \mathbb{Z}^{n-1}$. More information is carried by the next invariant,

$$Z_{0,2}(\chi) = \operatorname{coker} \Big(\mu_H \circ \bigwedge^2 \chi^\top : \bigwedge^2 G_2/G_3 \to \bigwedge^4 H \Big).$$

Set $Z(\mathcal{A}) := Z_{0,2}(\chi)$. It can be shown that $Z(\mathcal{A}) = \mathbb{Z}^{\binom{n-1}{3}-r} \oplus \mathbb{Z}_2^r$, where r is some integer that can be read off from the linking graph $\mathcal{G}$ of the link of $\mathcal{A}$.

For example, $Z(\mathcal{A}(1234)) = \mathbb{Z}$ and $Z(\mathcal{A}(2134)) = \mathbb{Z}_2$, showing again that the two arrangements have different cohomology rings. But $Z(\mathcal{A})$ is not a complete invariant of the cohomology ring. For example, $Z(\mathcal{A}(21435)) = Z(\mathcal{A}(31425)) = \mathbb{Z}_2^4$, although the two arrangements are distinguished by the ν-invariants (see below).

n	$\mathcal{A}$	$\nu_{3,0}$	$\nu_{3,1}$	$\nu_{3,2}$	$\nu_{3,3}$	$\nu_{3,4}$
3	$\mathcal{A}(123)$	9	4			
4	$\mathcal{A}(1234)$	27	0	13		
	$\mathcal{A}(2134)$	18	20	2		
5	$\mathcal{A}(12345)$	81	0	0	40	
	$\mathcal{A}(21345)$	54	27	35	5	
	$\mathcal{A}(21435)$	36	66	17	2	
	$\mathcal{A}(31425)$	51	60	10	0	
6	$\mathcal{A}(123456)$	243	0	0	0	121
	$\mathcal{A}(213456)$	162	81	0	107	14
	$\mathcal{A}(321456)$	162	0	162	32	8
	$\mathcal{A}(215436)$	108	126	87	38	5
	$\mathcal{A}(214356)$	108	108	121	24	3
	$\mathcal{A}(312546)$	72	186	90	14	2
	$\mathcal{A}(341256)$	81	162	112	6	3
	$\mathcal{A}(314256)$	117	162	74	10	1
	$\mathcal{A}(241536)$	108	200	48	8	0
	$\mathcal{L}$	81	162	112	6	3
	$\mathcal{M}$	144	160	60	0	0

Table 1. Arrangements of $n \leq 6$ planes in $\mathbb{R}^4$: Number $\nu_{3,d}$ of index 3 subgroups, according to their abelianization, $\mathbb{Z}^n \oplus \mathbb{Z}_3^d$.

6.7. Classification for $n \leq 6$

Let G the group of an arrangement of n transverse planes in $\mathbb{R}^4$, and G/G_3 its second nilpotent quotient. As can be seen in Table 1, the $\nu_{3,d}$-invariants completely classify the second nilpotent quotients (and, thereby the cohomology rings) of 2-arrangement groups, for $n \leq 6$, with a lone exception.

The exception is Mazurovskiĭ's pair, $\mathcal{K} = \mathcal{A}(341256)$ and $\mathcal{L}$. The corresponding configurations of skew lines in $\mathbb{R}^3$ were introduced in [24]. Explicit equations for $\mathcal{K}$ and $\mathcal{L}$ can be found in [23]. As noted in [24], the links of $\mathcal{K}$ and $\mathcal{L}$ have the same linking numbers. Thus, $H^*(X_{\mathcal{K}};\mathbb{Z}) \cong H^*(X_{\mathcal{L}};\mathbb{Z})$, and $G_{\mathcal{K}}/(G_{\mathcal{K}})_3 \cong G_{\mathcal{L}}/(G_{\mathcal{L}})_3$. On the other hand, $G_{\mathcal{K}}/(G_{\mathcal{K}})_4 \ncong G_{\mathcal{L}}/(G_{\mathcal{L}})_4$, as can be seen from the distribution of the abelianization of their index 3 subgroups, shown in Table 2.

d	0	1	2	3	4	5	6	7	8
$G_{\mathcal{K}}/(G_{\mathcal{K}})_4$	81	0	162	0	112	0	6	0	3
$G_{\mathcal{L}}/(G_{\mathcal{L}})_4$	81	0	172	24	78	6	0	0	3

Table 2. The groups $G_{\mathcal{K}}/(G_{\mathcal{K}})_4$ and $G_{\mathcal{L}}/(G_{\mathcal{L}})_4$: Number of index 3 subgroups, according to their abelianization, $\mathbb{Z}^6 \oplus \mathbb{Z}_3^d$.

We summarize the above discussion, as follows:

Theorem 6.8. *Let $(\mathcal{A}, \mathcal{A}') \neq (\mathcal{K}, \mathcal{L})$ be a pair of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$. Then $H^*(X) \cong H^*(X')$ if and only if $X \simeq X'$.*

In other words, up to 6 planes, and with the exception of Mazurovskiĭ's pair, the classification of complements of 2-arrangements up to cohomology-ring isomorphism coincides with the homotopy-type classification. As shown in [23], the latter coincides with the isotopy-type classification, modulo mirror images.

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