MTH3483 — TOPICS IN TOPOLOGY — FALL 1998

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Homework 3

1. Let ξ be a k-plane bundle with transition functions $\{\phi_{ij}\}$. Show:
   (a) If $k = 1$, then $\xi^{\otimes m} := \xi \otimes \cdots \otimes \xi (m \text{ factors})$ has transition functions $\{\phi_{ij}^m\}$.
   (b) The *determinant bundle* $\det \xi := \wedge^k \xi$ has transition functions $\{\det \phi_{ij}\}$.
   (c) $\det(\mathrm{T}^*(G_k(\mathbb{R}^n))) \cong (\det \gamma_k(\mathbb{R}^n))^{\otimes n}$.

2. Let γ be the tautological complex line bundle over $\mathbb{CP}^n$.
   (a) Find the transition functions of γ with respect to the usual cover of $\mathbb{CP}^n$.
   (b) Show: $\{\text{sections of } \gamma^m\} \cong \{\text{homogeneous polynomials in } \mathbb{C}^{n+1} \text{ of degree } m\}$.

3. Let $\widetilde{G}_k(\mathbb{R}^n)$ be the Grasmannian of *oriented* k-planes in $\mathbb{R}^n$.
   (a) There is a double covering $\mathbb{Z}_2 \to \widetilde{G}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$.
   (b) There is a principal bundle $\mathrm{SO}(k) \times \mathrm{SO}(n-k) \to \mathrm{SO}(n) \to \widetilde{G}_k(\mathbb{R}^n)$.
   (c) Interpret the covering map $G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ from part (a) as a map of homogeneous spaces, $\mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k) \to \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k)$.
   (d) Show that $\widetilde{G}_2(\mathbb{R}^4)$ is diffeomorphic to $S^2 \times S^2$.
   (e) What is the involution on $S^2 \times S^2$ that corresponds to the non-trivial covering map $\widetilde{G}_2(\mathbb{R}^4) \to \widetilde{G}_2(\mathbb{R}^4)$?

4. Let X be an n-connected CW-complex and Y an m-connected CW-complex. Show that their join, $X * Y$, is $(n + m + 1)$-connected.

5. Show that the inclusion-induced map $\pi_i(\mathrm{O}(k) \to \pi_i(\mathrm{O}(k + 1))$ is an epimorphism for $i = k - 1$ and an isomorphism for $i < k - 1$. Deduce that $\pi_i(V_k(\mathbb{R}^n)) = 0$.

6. Let G be a Lie group and H a closed subgroup. Show:
   (a) If K is a closed subgroup of H, there is as an H-bundle $H/K \to G/K \to G/H$.
   (b) If K is also a normal subgroup of H, this in fact a principal H/K-bundle.
   (c) There is an H-bundle $G/H \to BH \to BG$.

7. Let ξ be a principal G-bundle over B with classifying map $f : B \to BG$. Let H be a closed subgroup of G, and $i : H \to G$ the inclusion. Show:
$$\xi \text{ reduces to } H \iff \exists\, g : B \to BH \text{ such that } Bi \circ g \sim f.$$