MTH3483 — TOPICS IN TOPOLOGY — FALL 1998

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Homework 2

1. Let ξ be a principal G-bundle. Show:
   (a) $f^*(\xi)$ is a principal G-bundle.
   (b) If G acts effectively on F, then $f^*(\xi)[F] \cong f^*(\xi[F])$.
2. Given a principal G-bundle $P \to B$ and a closed subgroup H of G, show:
   (a) $P/H \to B$ is a G-bundle with fiber G/H associated to $P \to B$.
   (b) If H is also a normal subgroup of G, then $P/H \to B$ is a principal G/H-bundle.
3. Let G be a topological group acting on a space X. Show:
   (a) If $A \subset X$ is open, then $G \cdot A$ is open.
   (b) If G is compact and $A \subset X$ is closed, then $G \cdot A$ is closed.
   (c) If G is compact and $A \subset X$ is compact, then $G \cdot A$ is compact.
   (d) If X is Hausdorff, then X^G is Hausdorff.
   (e) If G is compact and X is Hausdorff, then X/G is Hausdorff.
   (f) The compactness assumption in part (e) is necessary.
   [Hint: Find an $\mathbb{R}$-action on $\mathbb{R}^2$ such that $\mathbb{R}^2/\mathbb{R}$ is not Hausdorff.]
4. Let ξ_n be the principal $\mathbb{Z}_n$-bundle $p_n : S^1 \to S^1$, where $p_n(z) = z^n$. Consider the associated $\mathbb{Z}_n$-bundle $\eta_n = \xi_n[S^1]$, where $\mathbb{Z}_n \subset S^1$ acts on S^1 by left-translation. Show that η_n is not trivial as a $\mathbb{Z}_n$-bundle, but it is trivial as a (principal) S^1-bundle.
5. Consider the $\mathbb{Z}_n$-action on S^3 given by $(z_1, z_2) \mapsto (\xi z_1, \xi z_2)$, where $\xi = e^{2\pi i/n}$. Let L_n be the orbit space.
   (a) Define a principal S^1-bundle $L_n \to S^2$.
   (b) What is the clutching function of this bundle?
   (c) Show that $L_1 = S^3$ and $L_2 = \mathrm{SO}(3)$.
6. Let $p : E \to B$ be a real vector bundle of rank k. Let
   $$F(E) = \{(b, \mathbf{f}) \mid b \in B \text{ and } \mathbf{f} = (f_1, \dots, f_n) \text{ is a basis for } p^{-1}(b)\}$$
   be the *space of frames* of E, and let $q : F(E) \to B$ be given by $q(b, \mathbf{f}) = b$. Show that:
   (a) $q : F(E) \to B$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.
   (b) The given vector bundle is associated to its frame bundle via the natural action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$, i.e., $E = F(E) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$.