# BUNDLES

1.1. Definition. A *bundle* $(E, B, p)$ consists of a *total space* $E$, a *base space* $B$, and a *projection map* $p : E \to B$ such that: for every $b \in B$, there exists an open neighborhood $U$ of $b$, a space $F$, and a homeomorphism $\phi : U \times F \to p^{-1}(U)$ such that the following diagram commutes:

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi} & p^{-1}(U) \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{\mathrm{id}} & U \end{array}$$

The condition above is referred to as *local triviality* of the bundle, and the pair $(U, \phi)$ as *local coordinates* about $b$.

Set $E_b := p^{-1}(b)$; this is called the *fiber over* $b$, and is identified to $F$ via $\phi : \{b\} \times F \xrightarrow{\sim} E_b$. If $b'$ is another point in $U$, we also have $\phi : \{b'\} \times F \xrightarrow{\sim} E_{b'}$, and so $E_b \approx E_{b'}$. Hence, if $B$ is path-connected (which we henceforth will allways assume), all the fibers are homeomorphic to $F$, the *typical fiber* of the bundle. We will often write the bundle as $F \to E \xrightarrow{p} B$ and say that $E$ fibers over $B$ with fiber $F$.

1.2. Remark. The map $p$ is onto. That's because $p^{-1}(b) \approx F \neq \emptyset$.

1.3. Remark. The map $p$ is open. To see this, it is enough to show that the restriction $p : p^{-1}(U) \to U$ is open, or, equivalently, $\mathrm{pr}_1 : U \times F \to U$ is open. But $\mathrm{pr}_1$ is the composite $U \times F \xrightarrow{\mathrm{id} \times \mathrm{c}} U \times * \xrightarrow{\sim} U$, where c is the map that collapses $F$ to a point. As c is obviously open, and the product of two open maps is open, we are done.

1.4. Definition. A *bundle morphism* $(u, f) : (E, B, p) \to (E', B', p)$ consists of maps $u : E \to E'$, $f : B \to B'$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

Note that $u$ maps the fiber over $b \in B$ to the fiber over $f(b) \in B'$. Hence the restriction of $u$ to $E_b$ defines a map

$$u_b : E_b \to E_{f(b)}.$$

Note also that $u$ determines $f$; indeed, $f(b) = p'(u(x))$, for any $x \in E_b$.

A morphism for which $B' = B$ and $f = \mathrm{id}$ is called a *B-morphism*. As the requirement is this case is $p' \circ u = p$, or $u(E_b) \subset E_b$, we also say that $u : E \to E'$ is a *fiber-preserving* map. If $u$ is injective, we say that $\xi = (E, B, p)$ is a *sub-bundle* of $\xi' = (E', B, p')$, and write $\xi \subset \xi'$.

A $B$-morphism $u : E \to E'$ which is a homeomorphism is called a *bundle isomorphism*; if such a morphism exists, we say the bundles $\xi = (E, B, p)$ and $\xi' = (E', B, p')$ are *equivalent*, and write $\xi \cong \xi'$. The automorphisms of a bundle $\xi$ are also called *gauge equivalences*; they form a group, $\mathcal{G}(\xi)$, called the *gauge group* of $\xi$.

The following lemma is a useful criterion for bundle equivalence.

1.5. Lemma. *Let $u : E \to E'$ be a $B$-morphism such that for each $b \in B, u_b : E_b \to E'_b$ is a homeomorphism. If the fiber $F$ is a locally connected, locally compact, Hausdorff space, then $u$ is a bundle isomorphism.*

Proof. Since the restriction of $u$ to any fiber is a bijection, $u$ itself is a bijection. All we have to prove is that the inverse of $u$ is continuous. It is enough to do that in a local coordinate chart.

So let $u : U \times F \to U \times F$ be a map given by $(b, x) \mapsto (b, u_b(x))$, where $u_b \in \mathrm{Homeo}(F)$. Endow $\mathrm{Homeo}(F)$ with the compact-open topology. Since $F$ is locally compact and Hausdorff, and since the map $(b, x) \mapsto u_b(x)$ is continuous, the map $\alpha : U \to \mathrm{Homeo}(F), b \mapsto u_b$ is also continuous (see [Bourbaki]). On the other hand, since $F$ is locally connected, locally compact and Hausdorff, the group structure on $\mathrm{Homeo}(F)$ (given by composition of maps) is compatible with the chosen topology (see [Bourbaki]). In particular, the map $\beta : \mathrm{Homeo}(F) \to \mathrm{Homeo}(F), g \mapsto g^{-1}$ is continuous. Thus $\beta \circ \alpha : U \to \mathrm{Homeo}(F), b \mapsto u_b^{-1}$ is continuous. This implies that $u^{-1} : U \times F \to U \times F, u^{-1}(b, x) = (b, u_b^{-1}(x))$ is also continuous, and we are done. □

1.6. Definition. A map $s : B \to E$ is called a *section* of the bundle $(E, B, p)$ if $p \circ s = \mathrm{id}_B$.

1.7. Definition. *A bundle $(B \times F, B, \mathrm{pr}_1)$ is called trivial. A bundle $(E, B, p)$ is trivializable if it is equivalent to a trivial one; a $B$-isomorphism $(E, B, p) \xrightarrow{\sim} (B \times F, B, \mathrm{pr}_1)$ is called a trivialization.*

A section of the trivial bundle $(B \times F, B, \mathrm{pr}_1)$ has the form $s : B \to B \times F, s(b) = (b, f(b))$. We thus have a bijection $\{\text{sections of trivial bundle}\} \longleftrightarrow \mathrm{Map}(B, F)$, given by $s \leftrightarrow f$.

1.8. Definition. Let $\xi = (E, B, p)$ be a bundle, and $f : B' \to B$ a map. The *pull-back* of $\xi$ by $f$ is the bundle $f^*(\xi) = (\widehat{E}, B', \hat{p})$, where $\widehat{E} = \{(b', x) \in B' \times E \mid f(b') = p(x)\}$ and $\hat{p}$ is the restriction of $\mathrm{pr}_1 : B' \times E \to B'$ to $\widehat{E}$.

If $\xi$ has local coordinates $(U, \phi)$, we may choose the local coordinates of $f^*(\xi)$ to be $(U', \phi')$, where $U' = f^{-1}(U)$ and $\phi' : U' \times F \to p'^{-1}(U')$ is given by $\phi'(b', y) = (b', \phi(f(b'), y))$.

There is a canonical morphism from $f^*(\xi)$ to $\xi$, given by the commuting square

$$\begin{array}{ccc} \widehat{E} & \xrightarrow{\hat{f}} & E \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

where $\hat{f}$ is the restriction of $\mathrm{pr}_2 : B' \times E \to E$ to $\widehat{E}$.

The pull-back has the following *universality property*: for every bundle morphism $(f', f) : (E', B', p') \to (E, B, p)$ there exists a unique $B'$-morphism $u : E' \to \widehat{E}$ such that $(f', f) = (\hat{f}, f) \circ (u, \mathrm{id}_{B'})$. The map $u$ is given by $u(x) = (p'(x), f'(x))$.

For each $b' \in B'$, the map $\hat{f}_{b'} : \widehat{E}_{b'} \to E_{f(b')}$ is a homeomorphism. Thus, if $\xi$ has fiber $F$, so does $f^*(\xi)$. The following theorem says that, under mild restrictions on $F$, these properties characterize pull-backs.

1.9. Theorem. *Let $(f', f)$ be a morphism from the bundle $\xi' = (E', B', p')$ to the bundle $\xi = (E, B, p)$ such that for each $b' \in B'$, $f'_{b'} : E'_{b'} \to E_{f(b')}$ is a homeomorphism. If the fiber $F$ of $\xi$ is a locally connected, locally compact, Hausdorff space, then $\xi'$ is equivalent to $f^*(\xi)$.*

Proof. By the universality property of pull-backs, there is a morphism $(u, \mathrm{id}_{B'})$ from $\xi'$ to $f^*(\xi)$ such that $f' = \hat{f} \circ u$. In particular, $f'_{b'} = \hat{f}_{b'} \circ u_{b'}$, and so $u_{b'}$ is a homeomorphism. Thus, by Lemma 1.5., $u : E' \to \widehat{E}$ is a bundle isomorphism. □

1.10. Definition. Let $\xi = (E, B, p)$ be a bundle, $A$ a subspace of $B$, and $i : A \to B$ the inclusion map. The *restriction* of $\xi$ to $A$ is the bundle $\xi|_A = i^*(\xi)$.

1.11. Definition. The *product* of the bundles $\xi_1 = (E_1, B_1, p_1)$ and $\xi_2 = (E_2, B_2, p_2)$ is the bundle $\xi_1 \times \xi_2 = (E_1 \times E_2, B_1 \times B_2, p_1 \times p_2)$.

If $\xi_i$ has local coordinates $(U_i, \phi_i)$, we may choose the local coordinates of $\xi_1 \times \xi_2$ to be $(U_1 \times U_2, \phi)$, where $\phi$ is the composite $U_1 \times U_2 \xrightarrow{\phi_1 \times \phi_2} (U_1 \times F_1) \times (U_2 \times F_2) \xrightarrow{\sim} U_1 \times U_2 \times F_1 \times F_2$. Note that the fiber of the product is the product of the fibers.

1.12. Definition. Let $\xi_1 = (E_1, B_1, p_1)$ and $\xi_2 = (E_2, B, p_2)$ be two bundles with the same base space. Their *Whitney sum* is the bundle $\xi_1 \oplus \xi_2 = \Delta^*(\xi_1 \times \xi_2)$, where $\Delta : B \to B \times B, \Delta(b) = (b, b)$ is the diagonal map.

Write $\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, B, p_1 \oplus p_2)$. We then have the following commuting square:

$$\begin{array}{ccc} E_1 \oplus E_2 & \xrightarrow{\widehat{\Delta}} & E_1 \times E_2 \\ {\scriptstyle p_1 \oplus p_2}\downarrow & & \downarrow{\scriptstyle p_1 \times p_2} \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

Note that $\xi_1$ and $\xi_2$ are sub-bundles of $\xi_1 \oplus \xi_2$, and that $F(\xi_1) \times F(\xi_2) = F(\xi_1 \oplus \xi_2)$. Under suitable restrictions on the fibers, these properties characterize Whitney sums:

1.13. Theorem. *Let $\xi_1$ and $\xi_2$ be sub-bundles of $\xi$ such that $F(\xi_1) \times F(\xi_2) = F(\xi)$. If $F(\xi)$ is a locally connected, locally compact, Hausdorff space, then $\xi'$ is equivalent to $\xi_1 \oplus \xi_2$.*

Proof. From the assumption and local triviality, we get a homeomorphism $u_b : (E_1)_b \times (E_2)_b \to E_b$, for every $b \in B$. This defines a morphism $u : E_1 \oplus E_2 \to E$ by $u(b, (x_1, x_2)) = (b, u_b(x_1, x_2))$. By Lemma 1.5, $u$ is a bundle isomorphism. □

**Exercises**

**1.** Show that $(f \circ g)^*(\xi) \cong g^*(f^*(\xi))$ and $\mathrm{id}^*(\xi) \cong \xi$.

**2.** If $\xi \cong \eta$, then $f^*(\xi) \cong f^*(\eta)$.

**3.** If $\xi$ is trivial, then $f^*(\xi)$ is trivial.

**4.** Let $\xi = (B \times F, B, \mathrm{pr}_1)$ be a trivial bundle. Show that $\mathcal{G}(\xi) \cong \mathrm{Map}(B, \mathrm{Homeo}(F))$.

**5.** Show that the restriction of the compact-open topology on $\mathrm{Homeo}(\mathbb{R}^n)$ to $\mathrm{GL}(n, \mathbb{R})$ coincides with the restriction of the Euclidean topology on $\mathbb{R}^{n^2}$ to $\mathrm{GL}(n, \mathbb{R})$.

**6.** Let $X = \{0\} \cup \{2^n\}_{n \in \mathbb{Z}}$ and consider the space $\mathrm{Homeo}(X)$, endowed with the compact-open topology. Show that $\beta : \mathrm{Homeo}(X) \to \mathrm{Homeo}(X), \beta(g) = g^{-1}$ is *not* continuous.

**7.** Consider the space $\mathrm{Homeo}(\mathbb{R}^2)$, endowed with the topology of pointwise convergence. Show that the map $\mathrm{Homeo}(\mathbb{R}^2) \times \mathrm{Homeo}(\mathbb{R}^2) \to \mathrm{Homeo}(\mathbb{R}^2), (u, v) \mapsto u \circ v$ is *not* continuous.