Northeastern University
Department of Mathematics

Prof. A. Suciu **MTH 3481 — TOPOLOGY 3** **Spring 1998**

Final Exam

This is a take-home exam, due Monday, June 15, at 9AM. Do 6 of the following 7 problems. Give complete proofs or justifications for each statement you make. Show all your work.

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1. Let $Y = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y = \sin\frac{1}{x}\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 0,\ -1 \le y \le 1\}$ and $X = \{0,1\}$. Let $f : X \to Y$, given by $f(0) = (0,0)$ and $f(1) = (\frac{1}{\pi}, 0)$.
   (a) Show that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism, for all $n \ge 0$.
   (b) Show that f is *not* a homotopy-equivalence.
   (c) Does this contradict Whitehead's theorem? Why, or why not?

2. Let X be a connected, finite CW-complex, with $\pi_1(X)$ having a non-trivial element of finite order. Let $Y = \widetilde{X} \times K(\pi_1(X), 1)$, where $\widetilde{X}$ is the universal cover of X.
   (a) Show that $\pi_n(X) \cong \pi_n(Y)$, for all $n \ge 0$.
   (b) Show that X is *not* homotopy-equivalent to Y.
   (c) Does this contradict Whitehead's theorem? Why, or why not?

3. Let $f : T^3 \to S^2$ be the composite of the Hopf bundle map $p : S^3 \to S^2$ and the quotient map $q : T^3 \to S^3$, which collapses the 2-skeleton of the 3-torus to a point.
   (a) Show that $f_* = 0 : \pi_n(T^3) \to \pi_n(S^2)$, for all $n \ge 0$.
   (b) Show that $f_* = 0 : H_n(T^3) \to H_n(S^2)$, for all $n > 0$.
   (c) And yet f is *not* homotopic to a constant map.

4. Let X be a connected CW-complex, with $\pi_i(X) = 0$ for $1 < i < n$, for some $n \ge 2$. Let $h : \pi_n(X) \to H_n(X)$ be the Hurewicz homomorphism. Show that $H_n(X)/h(\pi_n(X)) \cong H_n(K(\pi_1(X), 1)$.

5. Let G be an abelian group.
   (a) Show that $H_{n+1}(K(G,n)) = 0$, for $n > 1$.
   (b) Show that there is a Moore space $M(G,1)$ if and only if $H_2(K(G,1)) = 0$.
   (c) For what values of n does there exist a Moore space of type $M(\mathbb{Z}^n, 1)$?

6. Let $X = \mathbb{CP}^2 \cup e^3$, with attaching map $S^2 \xrightarrow{\times p} S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^2$, and $Y = M(\mathbb{Z}_p, 2) \vee S^4$.
   (a) Show that $M(\mathbb{Z}_p, 2)$ can be chosen so that X and Y have the same 3-skeleta.
   (b) Show that $H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z})$ (as graded rings).
   (c) Show that $H^*(X;\mathbb{Z}_p) \not\cong H^*(Y;\mathbb{Z}_p)$ (as graded rings).

7. Let G be a group, and let $\{M_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{Z}G$-modules.
   (a) Construct a CW-complex X with $\pi_1(X) = G$, and $\pi_n(X) = M_n$ (as $\mathbb{Z}G$-modules).
   (b) If $X = K(G,1) \times Y$, where $\pi_1(Y) = 0$, show that $\pi_n(X)$ is trivial as a $\mathbb{Z}G$-module, for all $n > 1$.
   (c) If $X = \mathbb{RP}^n$, show that $\pi_n(X) = \mathbb{Z}$ is trivial as a $\mathbb{Z}\mathbb{Z}_2$-module if and only if n is odd.