MTH3400 — GEOMETRY 1 — SPRING 1997

HOMEWORK #5

1. Prove that on any non-compact, connected differentiable manifold there exists an incomplete smooth vector field.

2. Give an example of a submersion $f : M \to N$, and a vector field $X \in \mathfrak{X}(M)$ which is not f-related to any $Y \in \mathfrak{X}(N)$.

3. Construct 3 linearly independent vector fields on S^3, and calculate their Lie brackets.

4. Let H be the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid x_{12}, x_{13}, x_{23} \in \mathbb{R} \right\}$$

This group has natural coordinates (x_{12}, x_{13}, x_{23}), and it acts on itself by left translations. Let $v_{12}, v_{13}, v_{,23}$ be the left-invariant vector-fields on H, with values at the identity $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively. Consider the 2-dimensional distributions E and F on H generated by v_{12}, v_{13} and v_{12}, v_{23}, respectively. Show that E is integrable and F is not.

5. Let $M = \mathbb{R}^3 \setminus \{0\}$. Consider the following 2-dimensional distributions on M:
(i) $E = \cup_{v \in M} E_v$, with $E_v = \{w \in T_vM \mid w \perp v\}$;
(ii) $F = \cup_{v \in M} F_v$, with $F_v = \{(a,b,c) \in T_vM \mid (b,c,a) \perp v\}$.
In each case, decide whether the distribution is integrable or not, and, if it is, describe the integral manifolds.