MTH3400 — GEOMETRY 1 — SPRING 1997

HOMEWORK #4

1. (i) Show that T^n is parallelizable, for all $n \geq 1$.
(ii) Show that $S^n \times S^k$ is parallelizable, for all n odd, and $k \geq 1$.
(iii) Show that S^{4n-1} has 3 independent vector fields, for all $n \geq 1$.

2. Let $X \in \mathfrak{X}(\mathbb{R})$ be the vector field

$$X = e^t \frac{d}{dt}.$$

For $a \in \mathbb{R}$, compute the integral curve $s_a : (-\epsilon_a, \delta_a) \to \mathbb{R}$ to X through a. (Be sure to specify its domain.) Find the flow determined by X. Is X a complete vector field?

3. Let $X, Y \in \mathfrak{X}(\mathbb{R})$ be smooth vector fields given by

$$X = x^2 \cos^2(\phi(x)) \frac{d}{dx}$$
$$Y = x^2 \sin^2(\phi(x)) \frac{d}{dx}$$

where $\phi : \mathbb{R} \to [0, \frac{\pi}{2}]$ is a smooth map such that $\phi \equiv 0$ on $[0, \frac{1}{5}] \cup [\frac{4}{5}, 1]$, $\phi \equiv \frac{\pi}{2}$ on $[\frac{2}{5}, \frac{3}{5}]$, and $\phi(x+1) = \phi(x)$. (Such a map exists by the smooth Urysohn lemma.) Prove that:
(i) X and Y are complete vector fields.
(ii) $X + Y$ is not complete.

4. Let $\Phi : \mathbb{R} \times M \to M$ be a (global) flow. A subset $C \subset M$ is said to be Φ-*invariant* if $\Phi_t(x) \in C$, for all $x \in C$, $t \in \mathbb{R}$. Let $R_x = \{\Phi_t(x)\}_{t \in \mathbb{R}}$ be the flow line through x. Show that the closure $\overline{R_x}$ is a Φ-invariant set.

5. Given $A \in \mathrm{Mat}(n)$, view the right translation

$$R_A : \mathrm{GL}(n) \to \mathrm{Mat}(n)$$

as a vector field $R_A \in \mathfrak{X}(\mathrm{GL}(n))$. (The value of this vector field at $B \in \mathrm{GL}(n)$ is the matrix $BA \in \mathrm{Mat}(n) = T_B(\mathrm{GL}(n))$.) For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

find the (global) flow $\Phi : \mathbb{R} \times \mathrm{GL}(2) \to \mathrm{GL}(2)$ generated by R_A.

6. Problem 13 from Chapter 5 in Spivak's book.