MTH3400 — GEOMETRY 1 — SPRING 1997

HOMEWORK #3

1. Let $p : E \to M$ be a vector bundle.

(i) Show there is a map of vector bundles (over M) $f : M \times \mathbb{R}^n \to E$ which is surjective on each fiber, if and only if there are sections $s_1, \ldots, s_n$ of E over M such that, for each $x \in M$, $s_1(x), \ldots, s_n(x)$ span the fiber $p^{-1}(x)$.

(ii) Show there is a map of vector bundles (over M) $f : M \times \mathbb{R}^n \to E$ which is injective on each fiber, if and only if there are sections $s_1, \ldots, s_n$ of E over M such that, for each $x \in M$, $s_1(x), \ldots, s_n(x)$ are linearly independent in the fiber $p^{-1}(x)$.

(iii) Show that E is isomorphic to the trivial rank n bundle $M \times \mathbb{R}^n \to M$ if and only if there exist sections $s_1, \ldots, s_n$ of E over M such that, for each $x \in M$, $s_1(x), \ldots, s_n(x)$ form a basis for the fiber $p^{-1}(x)$.

2. (i) Let $u = x^2 - y^3$, $v = 3xy + y^2 - x^2$. For which (a, b) in $\mathbb{R}^2$ is there a neighborhood U of (a, b) such that $(U, (u, v))$ is a coordinate system?

(ii) For which real numbers c is the locus $y^2 - x(x-1)(x-c) = 0$ a submanifold of $\mathbb{R}^2$?

3. Let $\mathbb{RP}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}/x \sim -x$ be the real projective space, with the standard differentiable structure. Let $\Phi : \mathbb{RP}^2 \to \mathbb{R}^5$ be given by

$$\Phi([x]) = (x_1^2, x_2^2, x_1x_2, x_1x_3, x_2x_3)$$

Show that Φ is a well-defined, smooth embedding. Compute $\Phi_* : T(\mathbb{RP}^2) \to T(\mathbb{R}^5)$.

4. (i) Prove that the group SO(3) of orthogonal 3×3 matrices with determinant 1 is an embedded submanifold of the 9-dimensional Euclidean space of all 3×3 matrices.

(ii) Prove that SO(3) with the differentiable structure described above is diffeomorphic to the real projective space $\mathbb{RP}^3$ with the standard differentiable structure.

5. Describe the structure of differentiable manifold on the complex projective space $\mathbb{CP}^n$ by explicitly defining coordinate charts and calculating transition functions.

6. Let M be a differentiable manifold and $f : M \to M$ a diffeomorphism. Consider the direct product $M \times [0, 1]$ with the identification of pairs of points $(f(x), 0)$ and $(x, 1)$, for all $x \in M$. Let $M_f = M \times [0, 1]/\sim$ be the quotient space (called the *mapping torus* of f, or, the *suspension construction* on f.)

(i) Show that M_f possesses a natural structure of differentiable manifold.

(ii) Show that M_f is a locally trivial fiber bundle with base S^1 and fiber M.

(iii) Apply the suspension construction to the following three cases:

(a) $M = \mathbb{R}$, $f(x) = -x$.
(b) $M = S^1$, $f(z) = -z$.
(c) $M = S^1$, $f(z) = \overline{z}$.

Identify the resulting mapping tori. In which of the three cases is the bundle $M_f \to S^1$ trivial?