MTH3400 — GEOMETRY 1 — SPRING 1997

HOMEWORK #2

1. Let $p \in S^n \subset \mathbb{R}^{n+1}$ and consider the tangent space at p,

$$T_p(S^n) = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1} \text{ is smooth, } \alpha(0) = p,\ \mathrm{im}(\alpha) \subset S^n\}.$$

Prove that $T_p(S^n)$ is the linear subspace of $\mathbb{R}^{n+1} = T_p(\mathbb{R}^{n+1})$ consisting of all vectors $v \in \mathbb{R}^{n+1}$ such that $v \perp p$.

2. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open subsets, and $f : U \to V$ a smooth map.

(i) Suppose there is a smooth map $g : V \to \mathbb{R}$ of rank 1 such that $g \circ f \equiv 0$ on U. Show that the rank r of the Jacobian matrix Jf is everywhere $< m$.

(ii) Suppose that f does have constant rank $r < m$ on U. Show that, on some open subset $W \supset f(U)$, there is a smooth map $g : W \to \mathbb{R}$ of rank 1 such that $g \circ f \equiv 0$ on U.

3. (See also Problem 33, p. 83, in Spivak's book.) Consider the *orthogonal group*

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n) \mid A^\top \cdot A = I\},$$

where $A^\top$ denotes the transpose of A, and I is the identity $n \times n$ matrix. Consider the map $\Phi : \mathrm{GL}(n) \to \mathrm{GL}(n)$, defined by $\Phi(A) = A^\top \cdot A$. Prove:

(i) Show that $\mathrm{O}(n)$ is a compact subspace of $\mathrm{GL}(n)$.

(ii) Φ is smooth.

(iii) Relative to the standard identification $T_I(\mathrm{GL}(n)) = \mathrm{Mat}(n)$, the differential $d_I\Phi : T_I(\mathrm{GL}(n)) \to T_I(\mathrm{GL}(n))$ is given by:

$$d_I\Phi(A) = A^\top + A.$$

(iv) The map Φ has constant rank $\frac{n(n+1)}{2}$.

(v) using the above, conclude that the orthogonal group $\mathrm{O}(n) \subset \mathrm{GL}(n)$ is a smooth, compact submanifold of dimension $\frac{n(n-1)}{2}$.

(vi) Show that the vector subspace $T_I(\mathrm{O}(n)) \subset \mathrm{Mat}(n)$ is the space of skew-symmetric matrices.

4. Problem 34, pp. 83-84, in Spivak's book.