MTH3400 — GEOMETRY 1 — SPRING 1997

HOMEWORK #1

1. Let X be a locally Euclidean space.
(i) Show that each component of X is an open subset of X.
(ii) Show that, if X is connected, then X is path-connected.

2. Let X and Y be connected, locally Euclidean spaces of the same dimension. If $f : X \to Y$ is bijective and continuous, show that f is a homeomorphism.

3. Let $\pi_\pm : S^n \setminus p_\pm \to \mathbb{R}^n$ be the stereographic projections. Write down explicit formulas for these maps, and prove that they are homeomorphisms.

4. Let $D^n = \{v \in \mathbb{R}^n \mid ||v|| \le 1\}$ be the unit n-disk, with boundary $\partial D^n = S^{n-1}$. Prove that D^n/S^{n-1} is homeomorphic to S^n.

5. Define an equivalence relation $\sim$ on $S^n = \{v \in \mathbb{R}^{n+1} \mid ||v|| = 1\}$ by writing $v \sim w$ if and only if $v = \pm w$. The quotient space $\mathbb{RP}^n = S^n/\sim$ is called the (real) projective n-space. Let $\pi : S^n \to \mathbb{RP}^n$, $\pi(v) = \{\pm v\}$ be the canonical projection. For each i, $1 \le i \le n+1$, define a subset U_i of $\mathbb{RP}^n$ by

$$U_i = \{\pi(x^1, \dots, x^{n+1}) \mid x^i \ne 0\}.$$

Prove the following facts, which together show that $\mathbb{RP}^n$ is an n-manifold.
(i) U_i is open in $\mathbb{RP}^n$.
(ii) $\{U_1, \dots, U_{n+1}\}$ covers $\mathbb{RP}^n$.
(iii) There is a homeomorphism $\phi_i : U_i \to \mathbb{R}^n$.
(iv) $\mathbb{RP}^n$ is compact, connected, and Hausdorff.

6. Let X be an n-dimensional manifold with boundary. Prove:
(i) $\partial X \cap \mathrm{int}(X) = \emptyset$.
(ii) ∂X is an $(n-1)$-manifold.
(iii) $\mathrm{int}X$ is an n-manifold.

7. Let M, N be manifolds with boundary. Prove that $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.

8. Let V and W be real vector spaces. Show how a choice of bases $e_1, \dots, e_m$ for V and $f_1, \dots, f_n$ for W defines an isomorphism of $\mathrm{Hom}_{\mathbb{R}}(V, W)$ with the real vector space of $n \times m$ matrices with real coefficients, $\mathrm{Mat}_{n\times m}(\mathbb{R})$. Shows that this isomorphism transforms composition

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) \times \mathrm{Hom}_{\mathbb{R}}(T, V) &\to \mathrm{Hom}_{\mathbb{R}}(T, V) \\ (f, g) &\mapsto f \circ g \end{aligned}$$

into matrix multiplication

$$\begin{aligned} \mathrm{Mat}_{n\times m}(\mathbb{R}) \times \mathrm{Mat}_{m\times k}(\mathbb{R}) &\to \mathrm{Mat}_{n\times k}(\mathbb{R}) \\ (A, B) &\mapsto A \cdot B \end{aligned}$$