Prof. A. Suciu

MTH 1230 **LINEAR ALGEBRA** **Spring 1999**

QUIZ 2

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1. $\boxed{\text{10 points}}$ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 8 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of A.

(b) What is the rank of A?

(c) Use the answer in part (a) to solve the system $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

2. $\boxed{\text{8 points}}$ Let

$$A = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

Decide whether the following products are defined or not. If they are, compute them:

$$A \cdot B, \quad B \cdot A, \quad A \cdot C, \quad C \cdot A, \quad B \cdot C, \quad C \cdot B.$$

3. $\boxed{\text{4 points}}$ For two invertible $n \times n$ matrices A and B, determine which of the following formulas are necessarily true:

(a) $(A - B)^2 = A^2 - 2AB + B^2$.

(b) $(I_n + A)(I_n - A + A^2) = I_n + A^3$.

(c) $(A^{-1}B^{-1}A)^{-1} = A^{-1}BA$.

(d) $ABA^{-1}B^{-1} = I_n$.

4. $\boxed{\text{8 points}}$ Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\vec{x}) = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} \vec{x}.$$

(a) Sketch the image of the unit square under this transformation.

(b) Explain how T can be interpreted as a reflection followed by a dilation. What is the angle of reflection, and what is the factor of dilation?

Geometric interpretation of some linear transformations

1. Rotation-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

geometrically, write the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates: $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Thus, T is counterclockwise rotation through the angle α, followed by a dilation by the factor r.

2. Projection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & b^2/a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = \frac{r}{\cos\alpha}\begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}.$$

Thus, T is orthogonal projection onto the line L making angle α with the x-axis, followed by a dilation by the factor $\frac{r}{\cos\alpha}$.

3. Reflection-dilations

To interpret the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \vec{x}$$

geometrically, write $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos 2\alpha \\ r\sin 2\alpha \end{bmatrix}$. Then the matrix of T can be written as

$$A = r\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

Thus, T is reflection in the line L making angle α with the x-axis, followed by a dilation by the factor r.