1	2	3	4	5	6	7	8	Σ

Name: ______________________

NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH 1230 **Prof. A. Suciu** **Spring 1999**

FINAL EXAM

Instructions: Put your name in the blanks above. Put your final answers to each question in the designated spaces—you may lose credit if you don't. Calculators are permitted. A single sheet of formulas is allowed. **Show your work.** If there is not enough room to show your work, use the back of the preceding page. Good luck!

---

1. [12 points] Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Are the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ linearly independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

2. 14 points The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & 3 & 5 & 2 & 1 \\ 3 & 5 & 8 & 3 & 1 \\ 4 & 7 & 11 & 4 & 1 \end{bmatrix}$ has the matrix $E = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) Compute:

- $\operatorname{rank} A =$
- $\dim(\operatorname{im} A) =$
- $\dim(\ker A) =$
- $\dim(\operatorname{im} A^\top) =$
- $\dim(\ker A^\top) =$

3. 12 points In each of the following cases, determine whether or not the given subset V of $\mathbb{R}^n$ is a vector subspace. If it is, identify it as either the kernel or the image of a matrix A, and write down the matrix A. If it is not a vector subspace, explain why not.

(a) $V = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \text{ where } t \text{ and } s \text{ take all real values}\}$

(b) $V = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \text{ where } t \text{ takes all real values}\}$

(c) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 x_4 = 0\}$

(d) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + 3x_3 = 0, \quad x_3 - x_4 = 0, \quad 2x_1 + x_3 + x_4 = 0\}$

4. 14 pts Let $A = \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$.

(a) Find the least squares solution $\begin{bmatrix} x^* \\ y^* \end{bmatrix}$ of the inconsistent system $A \cdot \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \vec{b}$.

(b) Find the 4×4 matrix associated with the projection of $\mathbb{R}^4$ onto the subspace $\operatorname{im} A$.

(c) Find the projection of $\vec{b}$ onto $\operatorname{im} A$.

5. 14 pts Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

and let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$.

(a) Apply the Gram-Schmidt process to the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,$ and write the result in the form $A = QR$.

(b) Compute the volume of the parallelepiped spanned by the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

6. 10 pts Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is a counterclockwise rotation of 30° about the y-axis, followed by a dilation by a factor of 6.

(a) Find the matrix A corresponding to T.

(b) What is the image of the vector $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ under the map T?

7. 10 points A 5×5 matrix A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 = 4$.

(a) Compute: $\operatorname{tr} A =$

(b) Compute: $\det A =$

(c) Compute: $\det(3I_5 - A) =$

(d) Is A invertible? Why, or why not?

(e) Is A orthogonal? Why, or why not?

8. [14 pts] Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 9 & -5 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.