Prof. Alexandru Suciu

MTH 1230 **LINEAR ALGEBRA** **Spring 2001**

SOLUTIONS to EXAM 2

1. 12 points

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 5 & 8 \end{bmatrix} \longrightarrow \text{rref}\, A = \begin{bmatrix} 1 & 0 & 0 & -2 & -\frac{7}{2} \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

(a) Find a basis for the image of A.

Columns of A corresponding to the pivot columns of rref A:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Find a basis for the kernel of A.

Independent solution vectors to equation $(\text{rref}\, A) \cdot \vec{x} = \vec{0}$:

$$\begin{bmatrix} 7 \\ -10 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Find the rank and the nullity of A.

$$\text{rank}\, A = \dim(\text{im}\, A) = \#\{\text{basis vectors of im}\, A\} = 3$$
$$\text{nullity}\, A = \dim(\ker A) = \#\{\text{basis vectors of}\ker A\} = 2$$

2. 16 points Consider the folowing four vectors in $\mathbb{R}^4$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 7 \\ 4 \end{bmatrix}.$$

Also let A be the 4×4 matrix with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 3 \\ 2 & 1 & 7 & 4 \end{bmatrix} \longrightarrow \text{rref}\, A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\text{rref}\, A) \cdot \vec{x} = \vec{0} \quad \Longrightarrow \quad \vec{x} = t \cdot \begin{bmatrix} -2 \\ \frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}$$

(a) Are the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

rank $A = \#\{\text{pivot columns of rref } A\} = 3$, which is less than $n = \#\{\text{columns of } A\} = 4$. Thus, the column vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are (linearly) dependent. A linear dependence is given by any non-zero vector in $\ker A = \ker(\text{rref } A)$. E.g., picking $t = 3$ gives:

$$-6\vec{v}_1 + \vec{v}_2 - 5\vec{v}_3 + 3\vec{v}_4 = \vec{0}.$$

(b) Do the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ form a basis for $\mathbb{R}^4$? Explain your answer.

The vectors are *not* independent, thus, per force, they do not form a basis.

(c) Does the equation $A \cdot \vec{x} = \vec{0}$ only have the solution $\vec{x} = \vec{0}$, or does it have other solutions? Explain your answer.

Yes, it does, since any vector $\vec{x}$ in $\ker A$ is also a solution (or, the nullity of A is > 0).

(d) Does the equation $A \cdot \vec{x} = \vec{b}$ have a solution for every choice of $\vec{b}$ in $\mathbb{R}^4$? Explain your answer.

No, it doesn't, since the rank of A is less than 4 (the number of columns of A).

3. 10 points Let V be the subspace of $\mathbb{R}^3$ defined by the equation $x_1 + 2x_2 - 5x_3 = 0$.

(a) Find a basis for V.

The space $V = \ker \begin{bmatrix} 1 & 2 & -5 \end{bmatrix}$ (a plane in $\mathbb{R}^3$) has basis

$$\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $\ker T = \{\vec{0}\}$ and $\operatorname{im} T = V$. Describe T by its matrix A.

$$A = \begin{bmatrix} 5 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. 12 points In each of the following, a subset V of $\mathbb{R}^3$ is given. Circle one answer:

(a) $V = \left\{ \begin{bmatrix} x+y+z \\ x+z \\ y \end{bmatrix} \;\middle|\; x, y, z \text{ arbitrary constants} \right\}$

Is closed under addition: YES
Is closed under scalar multiplication: YES
Is a vector subspace of $\mathbb{R}^3$: YES

(b) $V = \left\{ \begin{bmatrix} x+y+z \\ x+z \\ y+1 \end{bmatrix} \;\middle|\; x, y, z \text{ arbitrary constants} \right\}$

Is closed under addition: NO
Is closed under scalar multiplication: NO
Is a vector subspace of $\mathbb{R}^3$: NO

(c) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \text{ positive integers} \right\}$

Is closed under addition: YES
Is closed under scalar multiplication: NO
Is a vector subspace of $\mathbb{R}^3$: NO

(d) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; xy \le 0 \right\}$

Is closed under addition: NO
Is closed under scalar multiplication: YES
Is a vector subspace of $\mathbb{R}^3$: NO