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Name: ______________________

Instructor: ______________________

# NORTHEASTERN UNIVERSITY
## Department of Mathematics

**MTH 1230** **FINAL EXAM** **Fall 1997**

**Instructions**: Put your name and the instructor's name in the blanks above. Put your final answers to each question in the designated spaces—you may lose credit if you don't. Calculators are permitted. A single cribsheet of formulas is allowed. **SHOW YOUR WORK.** If there is not enough room to show your work, use the back of the preceding page. Good luck!

1. $\boxed{\text{8 points}}$ Are the following vectors independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix}$$

2. $\boxed{\text{20 points}}$ Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 5 & 7 \\ -6 & 10 & -7 \end{bmatrix}$, $\quad b = \begin{bmatrix} -2 \\ -3 \\ 8 \end{bmatrix}$.

(a) Solve the system of linear equations $Ax = b$, indicating clearly the row operations, pivots, leading variables, and free variables.

(b) Find the $LU$-decomposition of the matrix $A$.

(c) Find a basis for each of the row space, column space, and null-space of $A$.

(d) Find the rank of $A$.

(e) Find a non-zero vector that is orthogonal to both the vectors $\begin{bmatrix} 2 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} -6 & 10 & -7 \end{bmatrix}$.

3. $\boxed{\text{12 points}}$ In each of the following, a vector space $V$ and a subset $S$ are given. Circle one answer:

(a) $V = \mathbb{R}^4$, $S = \{(-t, 4t, 3t, 0) \mid t \text{ is a real number}\}$

$S$ is closed under addition: YES NO

$S$ is closed under scalar multiplication: YES NO

$S$ is a vector subspace of $V$: YES NO

(b) $V = \mathbb{R}^4$, $S = \{(-t, 4t, 3t, 1) \mid t \text{ is a positive real number}\}$

$S$ is closed under addition: YES NO

$S$ is closed under scalar multiplication: YES NO

$S$ is a vector subspace of $V$: YES NO

(c) $V = \mathbb{R}^4$, $S = \{x \in V \mid x_1 + 2x_2 - x_3 = 0, \quad 2x_1 + x_3 - 3x_4 = 0\}$

$S$ is closed under addition: YES NO

$S$ is closed under scalar multiplication: YES NO

$S$ is a vector subspace of $V$: YES NO

(d) $V = \mathbb{R}^4$, $S = \{x \in V \mid x_1 + 2x_2 - x_3 \geq 0\}$

$S$ is closed under addition: YES NO

$S$ is closed under scalar multiplication: YES NO

$S$ is a vector subspace of $V$: YES NO

4. $\boxed{\text{12 pts}}$

(a) Find the $3 \times 3$ matrix $A$ associated with the linear mapping that rotates the $yz$-plane by $-60°$ and reflects the $x$-axis about the $yz$-plane.

(b) Is $A$ orthogonal?

(c) What is $\det(A)$?

(d) What is the image of the point $(-3, 2, 1)$ under the above mapping?

5. $\boxed{\text{12 pts}}$ Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 4 \end{bmatrix}$.

(a) Find the least squares solution $\left[\begin{smallmatrix} \overline{x} \\ \overline{y} \end{smallmatrix}\right]$ of the inconsistent system $A \cdot \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \mathbf{b}$.

(b) Use your answer to part (a) to find the projection of $\mathbf{b}$ onto $\mathrm{CS}(A)$, the column space of $A$.

6. $\boxed{16 \text{ pts}}$ Let $\quad \mathbf{a}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

(a) Find unit vectors in the direction of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively.

(b) Find the lengths of $\mathbf{a}_1$ and $\mathbf{a}_2$, and compute the dot product $\mathbf{a}_1 \cdot \mathbf{a}_2$.

(c) Find the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$. Are $\mathbf{a}_1$ and $\mathbf{a}_2$ orthogonal?

(d) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Use the Gram-Schmidt process to find the $QR$-factorization of $A$.

7. $\boxed{\text{20 pts}}$ Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$.

(a) Find the characteristic equation for $A$.

(b) Find the eigenvalues of $A$.

(c) Find a basis for each eigenspace of $A$.

(d) Form a matrix $S$ using the two independent eigenvectors from part (c) as column vectors, and calculate $S^{-1}$.

(e) Calculate $S^{-1} \cdot A \cdot S$. Explain your answer.