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MTH 1125 | Calculus 3 | Spring 2002

SOLUTIONS TO QUIZ 5

In each problem, decide whether the series converges or diverges. In either case, indicate which test you are using, and justify your answer carefully.

Problem 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$

- **Method 1: Comparison Test.** Compare the given series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n}}$, which (up to a constant factor of $\frac{1}{\sqrt{3}}$) is a p-series with $p = \frac{1}{2} \leq 1$, and thus diverges:
$$\frac{1}{\sqrt{3n-1}} > \frac{1}{\sqrt{3n}}, \qquad \text{since } \ 3n-1 < 3n$$
Hence, the given series also **diverges**.
- **Method 2: Integral Test.** Compute the corresponding improper integral:
$$\int_1^{\infty} \frac{1}{\sqrt{3x-1}} dx = \frac{1}{3} \int_2^{\infty} \frac{1}{\sqrt{u}} du = \frac{1}{3}\, 2\sqrt{u}\Big|_2^{\infty} = \frac{2}{3}\left(\lim_{u\to\infty} \sqrt{u} - \sqrt{2}\right) = \infty$$
Hence, the given series **diverges**.

Problem 2. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+1}$

- **Method 1: Comparison Test.** Compare the given series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}}$, which is a p-series with $p = 2 > 1$, and thus converges:
$$\frac{3}{n^2+1} > \frac{1}{n^2}, \qquad \text{since } \ 3n^2 > n^2 + 1$$
Hence, the given series also **converges**.
- **Method 2: Integral Test.** Compute the corresponding improper integral:
$$\int_1^{\infty} \frac{3}{x^2+1} dx = 3 \tan^{-1}(x)\Big|_1^{\infty} = 3\left(\lim_{x\to\infty} \tan^{-1}(x) - \tan^{-1}(1)\right) = 3\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{3\pi}{4} < \infty$$
Hence, the given series **converges**.

Problem 3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

Use the Ratio Test:
$$\rho = \lim_{n\to\infty} \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{100^n}{100^{n+1}} = \lim_{n\to\infty} \frac{n+1}{100} = \infty > 1$$
Hence, the given series **diverges**.

Problem 4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^n}{(3n+1)^n}$

Use the Root Test:
$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{2^n n^n}{(3n+1)^n}} = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} < 1$$
Hence, the given series **converges**.