Lecture 4

1. $S^l = \{ H_1, \ldots, H_n \}$, $H_i \subseteq \mathbb{C}$

Fix $K = \text{a comm. ring}$

Let $E$ be the exterior algebra over $K$ with generators $e_1, \ldots, e_n$; $\deg e_i = 1$

$E = \bigoplus_{i=0}^{\infty} E_i$ (as free $K$-module) $\deg E_i = i$

$E_0 = K$, $E_i = \bigoplus_{j=1}^{n} K e_j$, $E_p = \bigwedge E_i$, $e_i e_j = -e_j e_i$

$E_p = \bigoplus_{|s|=p} K e_{i_1} \cdots e_{i_p}$, $1 \leq i_1 < \cdots < i_p < s = \{1, \ldots, n\}$

Define $d : E \to E$, $K$-linear, $d^2 = 0$, $\deg d = -1$

$d : E_p \to E_{p-1}$, $d(ab) = d(a)b + (-1)^{\deg a} a d(b)$

if $a$ is homogeneous.

$de_i = 1$, $\forall i$

$I^+$ implies $de_5 = \sum (-1)^{|s|} e_{s_j}$, $s_j = s \backslash \{i, j\}$

$I(R) = \left\langle \left( de_5 \right)_S \right| S = \text{minimal dependent set} \right\rangle$

\text{circuits}

Ex: $x, y, z, x - y, x - z, y - z$

$e_1, \ldots, e_6$

Dep. sets: $124, 135, 236, 456, 1256, \ldots$
get as a generator of $I$
\[ e_1e_2 - e_1e_4 + e_2e_4 = (e_1 - e_2)(e_1 - e_4) \]
\[ e_1e_5 - e_1e_5 + e_3e_5 \]
\[ e_2e_3 - e_2e_6 + e_3e_6 \]
\[ e_4e_5 - e_4e_6 + e_5e_6 \]

P1: Show that these 4 generate $I(\mathbb{R})$.

\[ A = A(\mathbb{R}) = \frac{E}{I(\mathbb{R})} \]

OS-alg. (Orlik-Solomon)

P2: $F \& S$ is dep. then $C_5 \in I$

DEPENDENCIES OF COMBINATORICS

$A(\mathbb{R})$ is defined by the matroid of $\mathbb{R}$.

= intersection lattice of $\mathbb{R}$ (= the set of all $H_i$ ordered opposite to inclusion).

Dimensions:

<table>
<thead>
<tr>
<th>$E$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>15</td>
<td>26</td>
<td>35</td>
<td>46</td>
<td>57</td>
<td></td>
</tr>
</tbody>
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| $A$ | 1 | 6 | 11 | 6 | 0 | 0 | 0 |

$H(\mathbb{R}, t) = \sum_{p=0}^{8} A_p \cdot (A_p)^T \cdot P^n = 1 + 6t + 11t^2 + 6t^3$

$= (1+t)(1+2t)(1+3t)^2$

The lattice

$r \in X_{\text{odd}}, x \in L$
A set of atoms is dependent iff

\[ k \leq |S| < |S| \]

a) \( A \) is graded by \( L \), \( \forall x \in L \)

\[ A_x = \langle e_s | vs = x \rangle_k \]

Prove \( A_x \) is well-defined.

\[ A = \bigoplus_{x \in L} A_x \quad A_p = \bigoplus_{x \in L \atop \pi k x = p} A_x \]

b) \( A \) is filtered by \( L \), \( A(x) = A(\delta x) \)

where \( \delta x = \{ \text{for } x \in L^3 \} \quad A \supset A(x) \)

Prove \( A \supset A(x) \).

filtration is defined by the grading

\[ A(x) = \bigoplus_{y \leq x} A_y \]

\( \text{Homological Interpretation:} \)

\( P = \text{poset (finite)} \)

The ordered complex:

- simplices are flags in \( P \)
- linearly ordered subset
Via this, \( P \) gets algebraic topology.

There are other complexes for \( P \) homotopy equivalent to this.

E.g., if \( P \) is a lattice, it is atomic complex (with highest and lowest pts deleted).

- Simplices are sets of atoms bounded from above but not by highest element.

**Proving homotopy equivalence of flag and atomic complexes for a lattice.**

We need the relative atomic complex

\( \Delta \) complex on atoms

\( \text{simplices on atoms} \)

\( \text{atomic complex} \)

\( \text{on atoms} \)

\( \text{of the chain complex} \) \( D \)

where \( D_P = \langle \Sigma c \Delta, 1 \rangle \)

\( \text{and} \)

\( d : D_P \rightarrow D_{P-1} \)

\( d(\sigma) = \sum (-1)^{i-1} \sigma_i \quad \sigma = \{ h_1, \ldots, h_n \} \)

\( d^2 = 0 \)

\( V \sigma_j = V \sigma \)
\[ D = \bigoplus_{x \in L} D_x \]

as chain complex

where \[ D_x = \langle \sigma \mid V(\sigma) = x \rangle \]

\[ D_x = \left( \Delta_x \right)_{\text{atomic complex}} \]

more precisely \( V_x \) we have exact:

\[ 0 \to D'(x) \to D''(x) \to D(x) \to 0 \]

\[ D'(x) = \text{atomic comp} \quad \text{on } A_x \]

\[ D''(x) = \text{simplex on } A_x \]

\[ \Rightarrow \quad H_p(D_x) = H_{p-2}(L_x) \]

\[ L_x = \{ y \in L \mid y \leq x \} \]

\[ \underbrace{\text{Folkman theorem}}_{(1966)} \]

For \( P \)-lattice

\[ H_p(D_x) = \begin{cases} 0 & \text{codim} x \not\in P \\ K^{M_x} & \text{codim} x \in P \end{cases} \]

where \( M_x = M_2(x) \)