Homework Problems, second installment

1. (a) Show that the OS ideal for the rank-three braid arrangement with defining polynomial \( Q(A) = xyz(x - y)(x - z)(y - z) \) is generated by quadratic relations.
   
   (b) Find an arrangement for which the OS ideal is not generated by quadratic relations.

2. Show that \( e_S \) is in the OS ideal if \( S \) is dependent.

3. Show that the grading of the exterior algebra \( E \) by \( L \) given by \( E_X = \langle e_S \mid \bigvee S = X \rangle \) for \( X \in L \) induces a well-defined grading of the OS algebra \( A \).

4. Show the natural map \( A(A_X) \to A(A) \) is injective for every \( X \in L \).

5. Show that the atomic complex of \( L \) (whose simplices are sets of atoms whose join not equal to \( 1_L \)) is homotopy equivalent to the flag complex of \( L - \{0_L, 1_L\} \).

6. Let \( A \) be the braid arrangement in \( \mathbb{R}^\ell+1 \):
   
   \[ A = \{ \ker(x_i - x_j) \mid 1 \leq i < j \leq \ell + 1 \} \]
   
   Find \( T := T(A) \) and \( \mu(T) \) when \( \ell = 3 \). (Remark. If you set
   
   \[ x = x_1 - x_2, \ y = x_1 - x_3, \ z = x_1 - x_4 \]
   
   then \( A = \{ \ker(x), \ker(y), \ker(z), \ker(x - y), \ker(y - z), \ker(x - z) \} \)

7. Prove that \( W(A) \) is a finite reflection group if \( A \) is a reflection arrangement.

8. Which is the more obvious?
   
   (a) \( A(W(A)) = A \) for any reflection arrangement \( A \),
   
   (b) \( W(A(W)) = W \) for any finite reflection group \( W \)
   
   Prove the more obvious one and discuss the other one.

9. Let \( A = A_\ell \).
   
   (a) Show that \( A \) is a reflection arrangement.
   
   (b) Find \( |A| \) and \( |W(A)| \).
   
   (c) Describe the group \( W(A) \) or find a group isomorphic to \( W(A) \).
10. Let $\mathcal{A} = B_\ell$.
   
   (a) Show that $\mathcal{A}$ is a reflection arrangement.

   (b) Find $|\mathcal{A}|$ and $|W(\mathcal{A})|$.

   (c) Describe the group $W(\mathcal{A})$ or find a group isomorphic to $W(\mathcal{A})$.

11. Let $\mathcal{A} = D_\ell$.
   
   (a) Show that $\mathcal{A}$ is a reflection arrangement.

   (b) Find $|\mathcal{A}|$ and $|W(\mathcal{A})|$.

   (c) Describe the group $W(\mathcal{A})$ or find a group isomorphic to $W(\mathcal{A})$.

12. Find a system of simple roots and the Coxeter diagram of $B_3$.

13. Let $\mathcal{A}$ be a reflection arrangement. Show $|\mu(X)| = |\{w \in W \mid \text{Fix}(w) = X\}|$ for $X \in L(\mathcal{A})$.

14. Let $V$ be an $\ell$-dimensional vector space with a basis $e_i$ ($1 \leq i \leq \ell$). Define the inner product on $V$ by

   $$(e_i, e_i) = 4/9, \quad (e_i, e_j) = -1/18 \quad (i \neq j)$$

   Let

   $$\mathcal{A}_\ell = \{\ker(x_i - x_j) \mid 1 \leq i < j \leq \ell\},$$

   $$\ker(x_i + x_j + x_k) \mid 1 \leq i < j < k \leq \ell\},$$

   $$\ker(x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} + x_{i_5} + x_{i_6}) \mid 1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq \ell\} \}.$$

   It is known that $\Delta := \{e_i - e_{i+1} \mid 1 \leq i < \ell\}, e_{\ell-2} + e_{\ell-1} + e_{\ell}\}$ is a system of simple roots. Let $E_6 = \mathcal{A}_6$ and $E_7 = \mathcal{A}_7$.

   (a) Find $|E_6|$ and $|E_7|$.

   (b) Find the Coxeter diagrams of $E_6$ and $E_7$.

   (c) Show $E_6$ and $E_7$ are reflection arrangement.