§1) Combinatorial Decomposition of $R'(G,R)$

$E$ = central arr. w/ underlying matroid $G$

$R$ = field

$\overline{\text{Def}}$: $R'(G,R) = \{ x \in R^n | \dim_{F} A'(A,x) \geq 1 \}$

(R not a field)

$R'(A,R) = \{ x \in R^n | \exists y \in R^7 \text{ s.t. } x \# y \text{ and } y \sim \alpha \}$

where $x \# y \iff$ all $2 \times 2$ minors of $\begin{bmatrix} x & y \end{bmatrix}$ vanish

Thus: $R'(A,R) = \bigcup_{\Gamma \in NP(G,R)} V'(\Gamma,R)$

$NP(G) = \text{set of neighborly partitions of submatroids of } G$

$\Gamma \in NP(G)$, a neighborly partition of $G' \subseteq G$

(wlog $G'$ has ground set $[m]$)

$K(\Gamma,R) = \{ x \in R^m | \lambda x = 0 \forall \lambda \text{ rank two flat } X \text{ not contained in a block of } \Gamma \}$

= kernel of pt-line incidence matrix
\[ NP(G,R) = \{ \gamma \in NP(G) \mid \dim_K K(\gamma,R) \geq 2 \} \]

\[ V' (\Gamma, R) = \{ \lambda \in K(\Gamma, R) \mid \exists m \in K(\Gamma, R) \text{ s.t.} \]
\[ \lambda = m s \quad \forall \text{ blocks } S \text{ of } \Gamma \]
\[ \det \left[ \begin{array}{cc} \lambda_i & m_i \\ \lambda_j & m_j \end{array} \right] = 0 \]

**Ex:**

\[ \begin{array}{c}
\text{has nullity } c \text{ of the} \\
\text{incidence matrix}
\end{array} \]

**5.2)** Line structure of \( V'(\Gamma, R) \)

Note: \( x \in V'(\Gamma, R), c \in \mathbb{R} - \{0\} \Rightarrow c x \in V'(\Gamma, R) \)

\[ V'(\Gamma, R) = \text{proj. image of } V'(\Gamma, R) \]

Note: \( x \in V'(\Gamma, R) \Rightarrow \exists m \in V'(\Gamma, R) \text{ s.t.} \)
\[ R x + R m \in V'(\Gamma, R) \]
\[ \text{RA + RM is a proj. line in } V'(\Gamma, R) \]
i.e. \( V'(\mathbb{R}) \) is "ruled by lines"

Call \( \lambda \ast \mu := \mathbb{R} \lambda + \mathbb{R} \mu \)

Note in addition:

If \( S \) is a block of \( \Gamma \), then the line \( \lambda \ast \mu \) intersects the subspace

\[
D_S = \{ M \in K(\mathbb{R}) \mid M^2 = 0, \eta_i = 0 \forall i \in S \}
\]

\[\text{Pf: By hypothesis } \lambda^S \parallel \eta^S. \text{ Then } \exists \alpha, \beta \in \mathbb{R} \text{ s.t. } \alpha \lambda^S + \beta \eta^S = 0\]

Set \( \eta = \alpha \lambda + \beta \eta \) then \( \overline{\eta} \in (\lambda \ast \mu) \cap D_S \)

Thus \( V'(\mathbb{R}) \) is the union of the lines in \( K(\mathbb{R}) \) which meet \( D_S \) for every block \( S \) of \( \Gamma \)

**Ex:**

[Diagram of three lines with labeled coordinates]

\[ K(\mathbb{R}^2) = \mathbb{R}^2 \Rightarrow \text{alg. closure} \]

\[ K(\mathbb{R}) = \mathbb{P}^2 \]

\[ d = \{ 0 \} \text{ non-zero on these } \Gamma = 1 2 3 4 5 6 7 \]
\[ V'(\Gamma, R) = \overline{K(\Gamma, R)} = \mathbb{P}^2 \]

Ex. Braid arr.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 4 & 5 \\
4 & 5 & 6 \\
\end{array}
\]

Incidence matrix is

\[
\begin{bmatrix}
4 & & \\
& 4 & \\
& & 4
\end{bmatrix}
\]

\[ \Gamma = 12\, 34\, 56 \]

\[ \Rightarrow K(\Gamma, R) \cong \mathbb{P}^2 \text{ for any } R \]

\[ \overline{V'(\Gamma, R)} \]

\[ D_{4}, D_{5} \]

00 11 11 11 00 00 11 11 11-1-1 11-1-1
\( R = \mathbb{Z}_2 \quad \overline{k(P, R)} = \mathbb{P}^3 \)

\( \overline{V'(P, R)} = \overline{k(P, R)} \)

Ex: \( P = \text{Hessian} \)

12 lines in \( \mathbb{P}^2 \)

\( G = \)

4 blocks of size 3

\( \dim k(P, R) = 3 \quad \text{if} \quad R = \mathbb{C} \quad \text{and} \quad \overline{V'(P, R)} = \overline{k(P, R)} \)