Combining Topology and Complement of the Complex.Topological Classification

$R: \approx r \in C^2, M = M(R) = C^2 - \cup R$

$L = \cap - lattice$

Then:
1. $H^*(M) \cong A(R)$ as rings
2. $A(R)$ is uniquely determined by $L(R)$

Question 1: Is the homotopy type or homeomorphism type of $M$ determined by $L(R)$?

Answer: No

Example:

$\pi_1(\bullet) = \mathbb{Z} \times \mathbb{Z} \cong \pi_1(\times)$

In fact, $M(\bullet) = M(\times)$

$R_1 = \bullet \quad R_2 = \times$

$Q_1 = (x+1)(x-1) \quad Q_2 = (x+y)(x-y) y$

$\text{c}R_1: \bigcirc \quad \text{c}R_2: \bigotimes$

$cQ_1 = (x+z)(x-z) y z \quad cQ_2 = (x+y)(x-y) y z$

$M(R_1) = M(d(cR_1)) = \frac{M(cR_1)}{T^*} \cong M(cR_2)$

$= M(d(cR_2)) = M(R_2)$
Question 2: Are there central arrangements for which $M(\sigma_1) \equiv M(\sigma_2)$ but $L(\sigma_1) \neq L(\sigma_2)$?

OR: Are there central arrangements for which the $A(\sigma_1) \equiv A(\sigma_2)$ but $L(\sigma_1) \neq L(\sigma_2)$?

Answer: Yes. So, instead we ask the following: What combinatorial features of $L$ can be extracted from $A$?

Observation 1: $A(\sigma)$ determines $\text{Poin}(L, t) = \sum_{\sigma \in S_n} |M(\sigma)| t^{|M(\sigma)|}$

Example: (3, 1, 3)

$\text{Poin}(L, t) = (1 + t)(1 + 3t^3)$

$= 1 + 7t + 15t^2 + 9t^3$

The coefficients of $\text{Poin}(L, t)$ are called "Whitney numbers of $L$ of the 2nd kind."

\[1 0 1 0 \]

\[1 0 0 0 0\]
supersolvable

\[ P_{\text{olv}}(1, \ell) = (1+\ell)(1+3\ell)^2 \]

The OS algos are not iso. since one is supersolv. and the other isn't.

Construction: An invariant of the ring structure of \( A(L) \).

\[ A = \frac{E}{I}, \quad A' = E^3, \quad I^2 = \ker(\Lambda^2(A) \to A^2) \]

\[ \Delta: E^1 \otimes I^2 \to E^3 = \Lambda^3(A') \]

\[ x \otimes r \mapsto x \Lambda r \]

Def: \( q_3 = \dim(\ker \Delta) \)

\[ \text{Ex}: \quad q_3 = 17 \quad \quad \# \quad q_3 = 12 \]

\[ I^2 = \langle \text{de}_3 \mid 3 \text{ dependent, } 151 = 3 \rangle \]

\[ = \langle \text{de}_{ijk} = (e_i - e_j) \wedge (e_i - e_k) \mid i, j, k \text{ dependent} \rangle \]

\[ \Rightarrow \]

\[ (e_i - e_j) \otimes \text{de}_{ijk} \in \ker \Delta \]

\[ (e_i - e_k) \otimes \text{de}_{ijk} \in \ker \Delta \]
Thm (Sullivan, Morgan: rational knot theory)

\( \Phi_3 = \text{rank of the 3rd factor in the lower central series of } \Pi_i \)

\[ \Pi = \Pi', \quad \lbrack \Pi, \Pi \rbrack = \Pi^2, \ldots, \lbrack \Pi, \Pi^{k+1} \rbrack = \Pi^{k+1} \]

\[ \Phi_3 = \text{rank} \left( \frac{\Pi^3}{\Pi^4} \right) \]

Thm (Falk, Randall) If \( \Omega \) is supersolvable, then ranks of factors in the LCSs of \( \Pi_i \) are determined by the exponents.

Ex: \( \Pi_1 (\#) = P_4 \)

\[ \Pi_1 (\#\#) = F_3 \times F_2 \times F_1 \]

Question: Is \( A(\Omega_1) \equiv A(\Omega_2) \)? Answer: No

Exercise: Show that \( L(1, \Omega_2) \) splits over \( \mathbb{Z} \) but \( \Omega_2 \) is not free.
Question: Are there arrs. \( R_1 \) and \( R_2 \) for which \( L(R_1) = L(R_2) \) but (i) \( M(R_1) \neq M(R_2) \)? Yes 
(ii) \( \pi_1(M(R_1)) \neq \pi_1(M(R_2)) \)? Yes

Projective equivalence

\[ R_2 = \{ H_1, ..., H_n \} \]

central \( \overset{\text{central}}{\xrightarrow{\text{central}}} \)

\[ \cap \ x_i \in V^* \quad \text{s.t.} \quad H_i = \ker \langle x_i \rangle \]

\[ B = B(R_2) = \begin{bmatrix} x_1 \\
\vdots \\
x_n \\
\end{bmatrix} \]

w/ basis \( x_1, ..., x_n \)

Then \( R_1 \) is projectively equivalent to \( R_2 \)
iff \( \exists \ M \times M \) non-singular matrix \( C \) and
\( N \times N \) non-singular matrix \( D \) s.t.

\[ B(R_2) = DB(R_1)C \]

If \( R_1 \) is projectively equ. to \( R_2 \) then

\[ M(R_1) \cong M(R_2) \]

Normal forms (for central arrs. of rank 3)

Assume \( H_1, H_2, H_3, H_4 \) are in "general position"

(so it will look like

\( \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \) (non \( \{ H_1, H_2, H_3 \} \)

are dependent)

Then \( R \) is projectively equ. to a unique arr. \( R_0 \) with

\[ B(R_0) = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \]
$\mathcal{R}_1$ is not proj.eq. to $\mathcal{R}_2$

but $\mathcal{L}(\mathcal{R}_1) \cong \mathcal{L}(\mathcal{R}_2)$.

Is $\mathcal{M}(\mathcal{R}_1) \cong \mathcal{M}(\mathcal{R}_2)$?

Randell's Lattice-isotopy theorem

Suppose $\mathcal{R}_2 = \{H_i(t), \ldots, \mathcal{H}_u(t)\}$

$0 \leq t \leq 1$ is a family of lattices satisfying $\text{codim} (\bigcap H_i(t))$ is constant for every $S$.

Then, if a continuous change of variables carrying $(C^\infty, \mathcal{U}(\mathcal{R}))$ to $(C^\infty, \mathcal{U}(\mathcal{R}_1))$ and $\mathcal{M}(\mathcal{R}_1)$ is diffeomorphic to $\mathcal{M}(\mathcal{R}_2)$. 

(Complex #)

We can slide 4 past the intersect to set isotopy of complexes.