Instructor: Prof. A. Suciu Name: ________________________

MTH U576 **Rings and Fields** Spring 2007

FINAL EXAM

(1) (12 points) Consider the ring

$$R = \mathbb{Z}_2 \times \mathbb{Z}_4 = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\},$$

with usual addition and multiplication.

(a) List all the invertible elements in R.

(b) List all the zero-divisors in R.

(c) List all the idempotents in R.

(d) Is R a commutative ring with unit?

(e) Is R a field?

(f) Let $S = \{(0,0),(0,1),(1,0),(1,1)\}$. Is S a subring of R?

(g) Let $S = \{(0,0),(0,1),(0,2),(0,3)\}$. Is S a subring of R?

(2) (10 points)

(a) Find the remainder when $f(x) = x^6 - 3x^4 + 5$ is divided by $g(x) = x - 2$ in $\mathbb{Q}[x]$.

(b) For what value(s) of k is $x + 1$ a factor of $x^4 + 2x^3 + 3x^2 + kx + 4$ in $\mathbb{Z}_7[x]$?

(3) (9 points) Let $f(x) = x^3 + 2x^2 + x + 1$, viewed as a polynomial in $\mathbb{Z}_p[x]$. Determine whether f is irreducible when:

(a) $p = 2$

(b) $p = 3$

(c) $p = 5$

(4) (10 points) Consider the polynomial

$$f(x) = 2x^4 + 3x^3 - 3x^2 - 5x - 6$$

(a) What are **all** the rational roots of f allowed by the Rational Root Test?

(b) Use the above information to factor f as a product of irreducible polynomials (over $\mathbb{Q}$).

(5) (12 points) Consider the polynomial

$$f(x) = x^5 - 5x^4 + 25x^2 - 10x + 5.$$

(a) Show that f is irreducible in $\mathbb{Q}[x]$.

(b) Show that the congruence-class ring $K = \mathbb{Q}[x]/(f(x))$ is a field.

(c) Is the extension $\mathbb{Q} \subset K$ algebraic? Why, or why not?

(d) Find a basis for K, viewed as a vector space over $\mathbb{Q}$.

(e) Compute $[K : \mathbb{Q}]$.

(6) (12 points) Consider the field $\mathbb{R}$, viewed as a vector space over $\mathbb{Q}$.

(a) Is the subset $\{1, \sqrt{3}\}$ linearly independent (over $\mathbb{Q}$)?

(b) Is $\sqrt{5}$ a linear combination of 1 and $\sqrt{3}$ (over $\mathbb{Q}$)?

(c) Does the subset $\{1, \sqrt{3}\}$ span $\mathbb{R}$ (as a vector space over $\mathbb{Q}$)?

(d) Find the minimal polynomial of $\sqrt{1+\sqrt{5}}$ over $\mathbb{Q}$.

(7) (9 points) Let $F \subset K$ be an extension of fields. Let $u \in K$ and $c \in F$.

(a) Suppose u is algebraic over F. Show that $u + c$ is algebraic over F.

(b) Suppose u is transcendental over F. Show that $u + c$ is transcendental over F.

(c) Show that $F(u) = F(u + c)$.

(8) (12 points) Let $p(x) = x^2 + bx + c$ be an **irreducible**, monic, quadratic polynomial in $\mathbb{Q}[x]$, and let $K = \mathbb{Q}[x]/(p(x))$.

(a) Show that K contains all the roots of $p(x)$.

(b) Is K a splitting field for p? Why, or why not?

(c) Is the extension $\mathbb{Q} \subset K$ a normal extension? Why, or why not?

(d) Is the extension $\mathbb{Q} \subset K$ a Galois extension? Why, or why not?

(e) What is the Galois group of p?

(9) (14 points) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ be the splitting field of $f(x) = (x^2 - 2)(x^2 - 5)$ over $\mathbb{Q}$.

(a) Find a basis of K, viewed as a vector space over $\mathbb{Q}$.

(b) What is a typical element of K, expressed in terms of this basis?

(c) What are the Galois automorphisms of the extension $\mathbb{Q} \subset K$? List them **all**, by indicating how they act on the basis found above (or, on the typical element of K).

(d) What is the Galois group $\mathrm{Gal}_{\mathbb{Q}}(K)$?

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(e) List **all** the subgroups of $\mathrm{Gal}_{\mathbb{Q}}(K)$.

(f) For each such subgroup H, indicate the corresponding fixed field E_H.

(g) Put together all this information by drawing a diagram of the Galois correspondence for the extension $\mathbb{Q} \subset K$.