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Homotopy

Two continuous functions from one topological space to another are called *homotopic* if one can be "continuously deformed" into the other, such a deformation being called a *homotopy* between the two functions. More precisely, we have the following definition.

Definition 1. Let X, Y be topological spaces, and $f, g\colon X \to Y$ continuous maps. A *homotopy* from f to g is a continuous function $F\colon X \times [0,1] \to Y$ satisfying

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x), \text{ for all } x \in X.$$

If such a homotopy exists, we say that f is *homotopic* to g, and denote this by $f \simeq g$.

If $g = \mathrm{const}_y$ is a constant map (i.e., there is a $y \in Y$ such that $g(x) = y$ for all $x \in X$), then we say that f is *nullhomotopic*.

Example 2. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ any two continuous, real functions. Then $f \simeq g$.

To see why this is the case, define a function $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(x,t) = (1-t)\cdot f(x) + t \cdot g(x).$$

Clearly, F is continuous, being a composite of continuous functions. Moreover, $F(x,0) = (1-0)\cdot f(x) + 0 \cdot g(x) = f(x)$, and $F(x,1) = (1-1)\cdot f(x) + 1 \cdot g(x) = g(x)$. Thus, F is a homotopy between f and g.

In particular, this shows that any continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is nullhomotopic.

This example can be generalized. First, we need a definition.

Definition 3. A subset $A \subset \mathbb{R}^n$ is said to be *convex* if, given any two points $x, y \in A$, the straight line segment from x to y is contained in A. In other words,

$$(1-t)x + ty \in A, \text{ for every } t \in [0,1].$$

Proposition 4. *Let A be a convex subset of $\mathbb{R}^n$, endowed with the subspace topology, and let X be any topological space. Then any two continuous maps $f, g\colon X \to A$ are homotopic.*

Proof. Use the same homotopy as in Example 2. Things work out, due to the convexity assumption. □

Let X, Y be two topological spaces, and let $\mathrm{Map}(X, Y)$ be the set of all continuous maps from X to Y.

Theorem 5. *Homotopy is an equivalence relation on* $\mathrm{Map}(X, Y)$.

Proof. We need to verify that $\simeq$ is reflexive, symmetric, and transitive.

Reflexivity ($f \simeq f$). The map $F\colon X \times I \to Y$, $F(x,t) = f(x)$ is a homotopy from f to f.

Symmetry ($f \simeq g \Rightarrow g \simeq f$). Suppose $F\colon X \times I \to Y$ is a homotopy from f to g. Then the map $G\colon X \times I \to Y$,

$$G(x,t) = F(x, 1-t)$$

is a homotopy from g to f.

Transitivity ($f \simeq g$ & $g \simeq h \Rightarrow f \simeq h$). Suppose $F\colon X \times I \to Y$ is a homotopy from f to g and $G\colon X \times I \to Y$ is a homotopy from g to h. Then the map $H\colon X \times I \to Y$,

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le 1/2 \\ G(x,2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is a homotopy from f to h, as is readily checked. □

We shall denote the equivalence class of a map $f\colon X \to Y$ by $[f]$, and the of all such homotopy classes by

$$[X, Y] = \mathrm{Map}(X, Y)/\simeq .$$

Example 6. Let A be a convex subset of $\mathbb{R}^n$, endowed with the subspace topology. From Proposition 4, we see that $[X, A]$ has exactly one element, for any topological space X.

Proposition 7. *Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ be continuous maps, and let $g \circ f, g' \circ f'\colon X \to Z$ be the respective composite maps. If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.*

Proof. Let $F\colon X \times I \to Y$ be a homotopy between f and f' and $G\colon Y \times I \to Z$ be a homotopy between g and g'. Define a map $H\colon X \times I \to Z$ by

$$H(x,t) = G(F(x,t),t).$$

Clearly, H is continuous, $H(x,0) = G(F(x,0),0) = G(f(x),0) = g(f(x))$, and $H(x,1) = G(F(x,1),1) = G(f'(x),1) = g'(f'(x))$. Thus, H is a homotopy between $g \circ f$ and $g' \circ f'$. □

As a consequence, composition of continuous maps defines a function

$$[X, Y] \times [Y, Z] \to [X, Z], \quad ([f],[g]) \mapsto [g \circ f].$$