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MTH U565 **TOPOLOGY** **Spring 2008**

Some solutions to Homework 2

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1. (**Problem 3.11.**) Let T be a set and B a collection of subsets (containing the empty set and T itself). Show that if B is closed under finite intersections, then the collection of all unions of sets in B forms a topology on T.

We know the following about the set B:

$$\text{If } B_1, \dots, B_n \text{ are elements of } B\text{, then } \bigcap_{i=1}^{n} B_i \text{ is also an element of } B. \tag{1}$$

Now let's consider the collection of all unions of sets in B, cal it A. A typical element in A is of the form:

$$U = \bigcup_{i \in I} B_i, \tag{2}$$

for some elements $B_i \in B$, with the union indexed by a set I.

Note first that B is a subset of A: every element B_i of B is clearly an element of A (simply take $U = B_i$ in formula (2)).

We need to verify that A satisfies the axioms of a topology.

- By assumption, $\emptyset$ and T are elements of B; thus, they both belong to A.
- We need to show: if $\{U_j\}_{j \in J}$, with $U_j \in A$, then $\bigcup_{j \in J} U_j \in A$.
  Write $U_j = \bigcup_{i \in I_j} B_{j,i}$. Then:
  $$\bigcup_{j \in J} U_j = \bigcup_{j \in J} \bigcup_{i \in I_j} B_{j,i}$$
  is a union of elements of B, and thus belongs to A.
- We need to show: if $U_1, \dots, U_n$, with $U_j \in A$, then $\bigcap_{j=1}^{n} U_j \in A$. For simplicity, we'll do the case $n = 2$ (the general case follows by induction on n). We have:
  $$U_1 \cap U_2 = \left(\bigcup_{i \in I} B_i \right) \cap \left(\bigcup_{j \in J} B_j \right) = \bigcup_{(i,j) \in I \times J} B_i \cap B_j.$$
  By property (1), $B_i \cap B_j \in B$; thus, $U_1 \cap U_2$ is a union of elements of B, and hence belongs to A.

2. (**Problem 3.6.**) Let $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the multiplication function $m(x,y) = xy$. Sketch the preimage of the open interval $(1,2)$ and show that this preimage is open.

The preimage $m^{-1}(1,2)$ is an (open) band between the hyperbolas $xy = 1$ and $xy = 2$:

$$m^{-1}(1,2) = \{(a,b) \in \mathbb{R}^2 \mid 1 < ab < 2\}.$$

Pick a point (a,b) in this region. Without loss of generality, we may assume $a > 0$ (and thus $b > 0$, and $1 < ab < 2$); the case $a < 0$ is done entirely similarly.

To show $m^{-1}(1,2)$ is open, we need to find a real number $\delta > 0$ so that the open disk $B_\delta(a,b)$ is entirely contained in $m^{-1}(1,2)$.

Take a point $(x, 1/x)$ with $x > 0$ on the hyperbola $xy = 1$; the distance square to (a,b) is

$$f_1(x) = (x-a)^2 + (1/x - b)^2.$$

Differentiating f_1 with respect to x, we find that the minimum distance occurs at the (unique) positive real root, call it λ_1, of the polynomial $g_1(x) = x^4 - ax^3 + bx - 1$. Plugging in, we find that the minimum distance from (a,b) to the hyperbola $xy = 1$ is:

$$\delta_1 = \sqrt{(\lambda_1 - a)^2 + (1/\lambda_1 - b)^2}.$$

Now take a point $(x, 2/x)$ with $x > 0$ on the hyperbola $xy = 2$; the distance square to (a,b) is

$$f_2(x) = (x-a)^2 + (2/x - b)^2.$$

Differentiating f_2 with respect to x, we find that the minimum distance occurs at the (unique) positive real root, call it λ_2, of the polynomial $g_2(x) = x^4 - ax^3 + 2bx - 4$. Plugging in, we find that the minimum distance from (a,b) to the hyperbola $xy = 2$ is:

$$\delta_2 = \sqrt{(\lambda_2 - a)^2 + (2/\lambda_2 - b)^2}.$$

Finally, set $\delta = \min(\delta_1, \delta_2)$. This is the required quantity, insuring that $B_\delta(a,b)$ is contained in $m^{-1}(1,2)$.