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MTH U565 **TOPOLOGY** **Spring 2008**

Handout 1: Inverse images and direct images

Let $f\colon A \longrightarrow B$ be a function, and let $U \subset B$ be a subset. The *inverse image* of U is the set $f^{-1}(U) \subset A$ consisting of all elements $a \in A$ such that $f(a) \in U$.

The inverse image commutes with all set operations: For any collection $\{U_i\}_{i\in I}$ of subsets of B, we have the following identities for

(1) Unions:

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i)$$

(2) Intersections:

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i)$$

and for any subsets U and V of B, we have identities for

(3) Complements:

$$\left(f^{-1}(U)\right)^{\complement} = f^{-1}(U^{\complement})$$

(4) Set differences:

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

(5) Symmetric differences:

$$f^{-1}(U \bigtriangleup V) = f^{-1}(U) \bigtriangleup f^{-1}(V)$$

In addition, for $X \subset A$ and $Y \subset B$, the inverse image satisfies the miscellaneous identities

(6) $(f|_X)^{-1}(Y) = X \cap f^{-1}(Y)$
(7) $f\left(f^{-1}(Y)\right) = Y \cap f(A)$
(8) $X \subset f^{-1}(f(X))$, with equality if f is injective.

Let $f\colon A \longrightarrow B$ be a function, and let $U \subset A$ be a subset. The *direct image* of U is the set $f(U) \subset B$ consisting of all elements of B which equal $f(u)$ for some $u \in U$.

Direct images satisfy the following properties:

(1) Unions: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f(U_i).$$

(2) Intersections: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f(U_i).$$

(3) Set difference: For any $U, V \subset A$,

$$f(V \setminus U) \supset f(V) \setminus f(U).$$

In particular, the complement of U satisfies $f(U^{\complement}) \supset f(A) \setminus f(U)$.

(4) Subsets: If $U \subset V \subset A$, then $f(U) \subset f(V) \subset B$.

(5) Inverse image of a direct image: For any $U \subset A$,

$$f^{-1}(f(U)) \supset U$$

with equality if f is injective.

(6) Direct image of an inverse image: For any $V \subset B$,

$$f(f^{-1}(V)) \subset V$$

with equality if f is surjective.