1. Let $f: X \to Y$ be a continuous surjection, and suppose $f$ is a closed map. Let $g: Y \to Z$ be a function so that $g \circ f: X \to Z$ is continuous. Show that $g$ is continuous.

**Proof.** It is enough to show: For every closed subset $F \subset Z$, the subset $g^{-1}(F)$ is closed.

Now, by continuity of $g \circ f$, we know that $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a closed subset of $X$. Since $f$ is a closed map, it takes this closed subset of $X$ to a closed subset of $Y$. But $f((g \circ f)^{-1}(F)) = f(f^{-1}(g^{-1}(F))) = g^{-1}(F)$, since $f$ is surjective. Hence, $g^{-1}(F)$ is closed. \qed

2. Let $X$ be a space. Show that $X$ is Hausdorff if, and only if, the diagonal $\Delta := \{(x, x) \mid x \in X\}$ is a closed subspace of $X \times X$.

**Proof.** Suppose $X$ is a Hausdorff space. We need to show that the complement of the diagonal, $\Delta^c := X \times X \setminus \Delta$, is open. So let $(x, y) \in \Delta^c$. Then $x \neq y$, and so there are disjoint open sets $U$ and $V$, containing $x$ and $y$, respectively. By definition of the product topology, $U \times V$ is an open subset of $X \times X$, and clearly $U \times V \subset \Delta^c$ (for otherwise $U \cap V \neq \emptyset$). This shows that $\Delta^c$ is open.

Conversely, suppose $\Delta$ is closed, that is to say, $\Delta^c$ is open. Let $x$ and $y$ be two distinct elements of $X$. Then $(x, y) \in \Delta^c$, and so there is a basis open set $U \times V \subset \Delta^c$ containing $(x, y)$. Now note that $U$ and $V$ are open, disjoint subsets of $X$, containing $x$ and $y$, respectively. This shows that $X$ is Hausdorff. \qed

3. Let $X = [0, 1]/(\frac{1}{4}, \frac{3}{4})$ be the quotient space of the unit interval, where the open interval $(\frac{1}{4}, \frac{3}{4})$ is identified to a single point. Show that $X$ is not a Hausdorff space.

**Proof.** Recall that in a quotient space $X/A = (X \setminus A) \coprod \{\ast\}$, the open sets are of one of two types:

1. either an open set in $X \setminus A$; or
2. of the form $\{\ast\} \cup (W \cap (X \setminus A))$, where $W$ is an open set in $X$, containing $A$.

In our situation, $X = [0, 1]$ and $A = (\frac{1}{4}, \frac{3}{4})$. Take $x = \frac{1}{4}$ and $y = \frac{3}{4}$, viewed as elements of $X/A$. Suppose $U$ and $V$ are open, disjoint subsets of $X/A$, containing $x$ and $y$, respectively. Then, necessarily, both $U$ and $V$ must be of type...
(2), since an open subset of $[0, 1]$ containing one of the endpoints of the interval $(\frac{1}{2}, \frac{3}{4})$ must intersect that interval. But then both $U$ and $V$ must contain the element $\{x\}$, and thus cannot be disjoint—a contradiction. □

4. Let $X$ be a Hausdorff space. Suppose $A$ is a compact subspace, and $x \in X \setminus A$. Show that there exist disjoint open sets $U$ and $V$ containing $A$ and $x$, respectively.

Proof. Let $y \in A$. Since $x \in X \setminus A$, we see that $y \neq x$. Since $X$ is Hausdorff, there are open, disjoint sets $U_y$ and $V_y$ containing $y$ and $x$, respectively.

Now note that $\{U_y\}_{y \in A}$ is an open cover of $A$. Since $A$ is compact, this cover admits a finite subcover, say, $U_{y_1}, \ldots, U_{y_n}$. Define:

$$U := \bigcup_{i=1}^{n} U_{y_i} \quad \text{and} \quad V := \bigcap_{i=1}^{n} V_{y_i}.$$ 

It is readily seen that $U$ and $V$ are the desired open sets. □

5. Let $p : X \to Y$ be a quotient map. Suppose $Y$ is connected, and, for each $y \in Y$, the subspace $p^{-1}\{y\}$ is connected. Show that $X$ is connected.

Proof. Suppose $X$ is disconnected, that is, there are disjoint, open, non-empty sets $U$ and $V$ such that $X = U \cup V$.

Consider the subsets $p(U)$ and $p(V)$ of $Y$: they are both open (since $U$ and $V$ are open, and $p$ is a quotient map), and non-empty (since $U$ and $V$ are non-empty). Thus, by the connectivity of $Y$, the sets $p(U)$ and $p(V)$ cannot be disjoint.

So let $y \in p(U) \cap p(V)$. We then have

$$p^{-1}\{y\} = \left(U \cap p^{-1}\{y\}\right) \cup \left(V \cap p^{-1}\{y\}\right).$$

Both sets on the right side are open subsets of $p^{-1}\{y\}$ (by definition of the subspace topology), and both are non-empty (since $y \in p(U)$ means $y = p(x)$, for some $x \in U$, and so $x \in U \cap p^{-1}\{y\}$, and similarly for the other subset). Thus, by the connectivity of $p^{-1}\{y\}$, these sets $U \cap p^{-1}\{y\}$ and $V \cap p^{-1}\{y\}$ cannot be disjoint. This means there is a $z \in U \cap V \cap p^{-1}\{y\}$. Consequently, $U \cap V \neq \emptyset$, a contradiction. □

6. Let $X$ be a discrete topological space, and let $\sim$ be an equivalence relation on $X$. Prove that $X/\sim$, endowed with the quotient topology, is also a discrete space.

Proof. Let $p : X \to X/\sim$ be the quotient map. By definition of quotient topology, a subset $U$ of $X/\sim$ is open if and only if $p^{-1}(U)$ is an open subset of $X$. But every subset of $X$ is open (since $X$ has the discrete topology). Hence, every subset of $X/\sim$ is open; that is to say, $X/\sim$ is discrete. □