Prof. Alexandru Suciu

MTH U371 **LINEAR ALGEBRA** **Spring 2006**

Solutions to Quiz 5

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1. 8 points Apply the Gram-Schmidt process to the vectors $\vec{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and write the result in the form $A = Q \cdot R$.

$$
\begin{aligned}
r_{11} = \|\vec{v}_1\| &= \sqrt{16+9} = 5 \\
\vec{u}_1 = \frac{1}{r_{11}}\vec{v}_1 &= \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
r_{12} = \vec{u}_1 \cdot \vec{v}_2 &= \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{2}{5} \\
\vec{v}_2^{\perp} = \vec{v}_2 - r_{12}\vec{u}_1 &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \frac{2}{5}\cdot\frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{11}{25}\begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
r_{22} = \|\vec{v}_2^{\perp}\| &= \frac{11}{25}\sqrt{9+16} = \frac{11}{5} \\
\vec{u}_2 = \frac{1}{r_{22}}\vec{v}_2^{\perp} &= \frac{5}{11}\cdot\frac{11}{25}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}
\end{aligned}
$$

$$A = Q \cdot R$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_1 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2/5 \\ 0 & 11/5 \end{bmatrix}$$

2. 6 points Consider the vectors $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix}$.

(a) Find the matrix of the orthogonal projection onto the line L in $\mathbb{R}^3$ spanned by $\vec{v}$.

$$
\begin{aligned}
\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} &= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\
A = \vec{u}\cdot\vec{u}^{\top} &= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}
\end{aligned}
$$

(b) Find the projection of $\vec{w}$ onto the line L.

$$\mathrm{proj}_L(\vec{w}) = A\vec{w} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

3. 5 points Is there an orthogonal transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$T\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}$ and $T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}$? Justify your answer, one way or the other.

An orthogonal transformation must preserve dot products of vectors:

$$T\vec{x} \cdot T\vec{y} = \vec{x} \cdot \vec{y}, \quad \text{for all vectors } \vec{x}, \vec{y}.$$

In this situation, though:

$$T\begin{bmatrix}1\\3\end{bmatrix} \cdot T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\-1\end{bmatrix} = 6 - 1 = 5$$

$$\begin{bmatrix}1\\3\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = 1 + 6 = 7$$

Thus, there is no such orthogonal transformation T.

4. 6 points Let A be an arbitrary $n \times n$ matrix. and let Q be an orthogonal $n \times n$ matrix. For each of the following questions, answer: "Yes, always," or "Sometimes yes, sometimes not," or "No, never." Justify your answer, as much as possible.

(a) The matrix $AA^\top$ is symmetric.

Yes, always: $(AA^\top)^\top = (A^\top)^\top A^\top = AA^\top$

(b) The matrix $AA^\top$ is invertible.

Sometimes yes, for example, when A is the identity matrix, and sometimes not, for example, when A is the zero matrix.

(c) The matrix $AA^\top$ is orthogonal.

Sometimes yes, for example, when A is the identity matrix, and sometimes not, for example, when A is the zero matrix.

(d) The matrix $Q^\top$ is symmetric.

Sometimes yes, for example, when Q is the identity matrix, and sometimes not, for example, when $Q = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$.

(e) The matrix $Q^\top$ is invertible.

Yes, always: $QQ^\top = I_n$, so Q is the inverse matrix to $Q^\top$.

(f) The matrix $Q^\top$ is orthogonal.

Yes, always: $Q^\top(Q^\top)^\top = Q^\top Q = I_n$.