| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Σ |
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Name: ______________________

**NORTHEASTERN UNIVERSITY**
**DEPARTMENT OF MATHEMATICS**

**MTH U371** **FINAL EXAM** **Spring 2006**

**Instructions**: Put your name in the blanks above. Put your final answers to each question in the designated spaces. Calculators are permitted. A single sheet of formulas is allowed. **Show your work.** If there is not enough room to show your work, use the back page.

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**1.** 12 points Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Are the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ linearly independent or dependent? If they are independent, explain why. If they are dependent, exhibit a linear dependence relation among them.

(b) Write the vector $\vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

**2.** 10 points The matrix $A = \begin{bmatrix} 1 & 0 & -1 & 2 & -3 \\ 4 & 1 & 6 & 3 & -2 \\ 1 & 1 & 9 & -3 & 7 \\ 3 & 1 & 7 & 1 & 1 \end{bmatrix}$ has $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 10 & -5 & 10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ as its row-reduced echelon form.

(a) Find a basis for the image of $A$.

(b) Find a basis for the kernel of $A$.

(c) Compute:

- rank $A =$
- $\dim(\ker A) =$
- $\dim(\text{im } A)^\perp =$
- $\dim(\ker A)^\perp =$

**3.** 12 pts

(a) Find the least squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

(b) Use your answer to part (a) to find the projection of $\vec{b}$ onto the image of $A$.

(c) Determine the error $||\vec{b} - A\vec{x}^*||$.

**4.** 12 pts Apply the Gram-Schmidt process to the vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

and write the result in the form $A = QR$, with $Q$ orthogonal and $R$ upper-diagonal.

**5.** 14 pts Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that rotates the $xy$-plane by $30°$ in a counterclockwise direction, and reflects the $z$-axis about the $xy$-plane.

(a) Find the matrix $A$ corresponding to $T$.

(b) Is $A$ orthogonal? Why, or why not?

(c) What is $\det(A)$?

(d) What is $\det(4A)$?

(e) Find $A^{-1}$.

(f) What is the image of the vector $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ under the map $T$?

**6.** 12 pts Let $A = \begin{bmatrix} 3 & 4 & 0 \\ 5 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(a) Find the characteristic polynomial of $A$.

(b) Find the eigenvalues of $A$.

(c) Find a basis for each eigenspace of $A$.

(d) Find an invertible matrix $S$ and a diagonal matrix $D$ such that $A = S \cdot D \cdot S^{-1}$.
[You do not have to calculate $S^{-1}$.]

**7.** 14 points A $2 \times 2$ matrix $A$ matrix has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$, with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Find $A$.

(b) Consider the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, with initial value $\vec{x}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Find a closed form for $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(c) What is $\vec{x}(4)$?

**8.** 14 pts Find the singular value decomposition for the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$, as follows:

(a) Find the symmetrized matrix $B = A^\top A$.

(b) Find the eigenvalues $\lambda_1$ and $\lambda_2$ of $B$.

(c) Find the singular values $\sigma_1$ and $\sigma_2$ of $A$.

(d) Find eigenvectors $\vec{w}_1$ and $\vec{w}_2$ for $B$.

(e) Now find an orthonormal set of eigenvectors, $\vec{v}_1$ and $\vec{v}_2$.

(f) Use the vectors $\vec{v}_i$, the singular values $\sigma_i$, and the matrix $A$ to find another orthonormal set, $\vec{u}_1$ and $\vec{u}_2$.

(g) Put everything together to arrive at the SVD decomposition, $A = U \cdot \Sigma \cdot V^\top$. Check your answer!