SOLUTIONS to QUIZ 3

1. 12 points Let $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 2 & 0 & 4 & -1 & 7 \\ -1 & 3 & 0 & 6 & 2 \end{bmatrix}$.

(a) Find the row reduced echelon form of A.

$$\text{rref}\, A = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2/3 & 0 & 11/3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

(b) Find a basis for the image of A.

Columns of A corresponding to the pivot columns of rref A:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix} \right\}.$$

(c) Find a basis for the kernel of A.

Independent solution vectors to equation $(\text{rref}\, A) \cdot \vec{x} = \vec{0}$:

$$\left\{ \begin{bmatrix} -3 \\ -11/3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(d) Find the rank and the nullity of A.

$$\text{rank}\, A = \dim(\text{im}\, A) = \#\{\text{basis vectors of im}\, A\} = 3$$
$$\text{nullity}\, A = \dim(\ker A) = \#\{\text{basis vectors of ker}\, A\} = 2$$

2. 10 points Consider the folowing four vectors in $\mathbb{R}^4$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 4 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

Let A be the 4×4 matrix with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 4 & -1 & 1 \\ -3 & 0 & -2 & -5 \\ 2 & -4 & 4 & 4 \end{bmatrix} \longrightarrow \text{rref}\, A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -7/2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\text{rref}\, A) \cdot \vec{x} = \vec{0} \quad \Longrightarrow \quad \vec{x} = t \cdot \begin{bmatrix} -5 \\ 7/2 \\ 5 \\ 1 \end{bmatrix}$$

(a) Are the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ independent or dependent? If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

rank $A = \#\{\text{pivot columns of rref } A\} = 3$, which is less than $\#\{\text{columns of } A\} = 4$.
Thus, the column vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ are **linearly dependent**.
A linear dependence is given by any non-zero vector in $\ker A = \ker(\text{rref } A)$.
For instance, picking $t = 1$ gives:

$$-5\vec{v}_1 + \frac{7}{2}\vec{v}_2 + 5\vec{v}_3 + \vec{v}_4 = \vec{0}, \quad \text{or} \quad \vec{v}_4 = 5\vec{v}_1 - \frac{7}{2}\vec{v}_2 - 5\vec{v}_3.$$

(b) Do the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ form a basis for $\mathbb{R}^4$? Explain your answer.

No. The vectors are *not* independent, thus, per force, they do not form a basis.

(c) Do the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ span $\mathbb{R}^4$? Explain your answer.

No. For, if 4 vectors span $\mathbb{R}^4$, they must form a basis. (Recall, a basis is a minimal spanning set, and $\dim \mathbb{R}^4 = 4$ is the number of vectors in any basis for $\mathbb{R}^4$; so, if the minimum number of vectors in a spanning set is attained, that set must form a basis.) By the preceding, we know this is not the case.

3. 8 points Let V be the subspace of $\mathbb{R}^3$ defined by the equation $2x_1 - 3x_2 + 4x_3 = 0$.

(a) Express V as the kernel of a matrix A.

$$V = \ker(A), \quad \text{where} \quad A = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}.$$

(b) Express V as the image of a matrix B.

$$V = \text{im}(B), \quad \text{where} \quad B = \begin{bmatrix} 3 & -2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Find a basis for V.

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$