Prof. Alexandru Suciu

MTH U371 **LINEAR ALGEBRA** **Spring 2005**

QUIZ 2

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1. $\boxed{\text{5 points}}$ Use Gaussian elimination to find the inverse of following matrix. Indicate for each step which row operation you use.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Compute:

$$[A \,\vdots\, I_3] \xrightarrow{\text{rref}} [I_3 \,\vdots\, A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 2 \\ 1 & 0 & -1 \\ 3 & -\frac{1}{2} & -2 \end{bmatrix}$$

2. $\boxed{\text{5 points}}$ For which choices of the constant k is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & k^2 \\ -3 & 0 & k \end{bmatrix}$$

Compute:

$$A \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & k^2 - 8 \\ 0 & 0 & -3k^2 + k + 30 \end{bmatrix}$$

Thus, A is invertible for all values of k, except those for which $-3k^2 + k + 30 = 0$; that is,

$$A \text{ is invertible } \iff k \neq -3 \text{ and } k \neq \frac{10}{3}.$$

3. $\boxed{\text{5 points}}$ Find the matrix A of the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad T \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

We have:

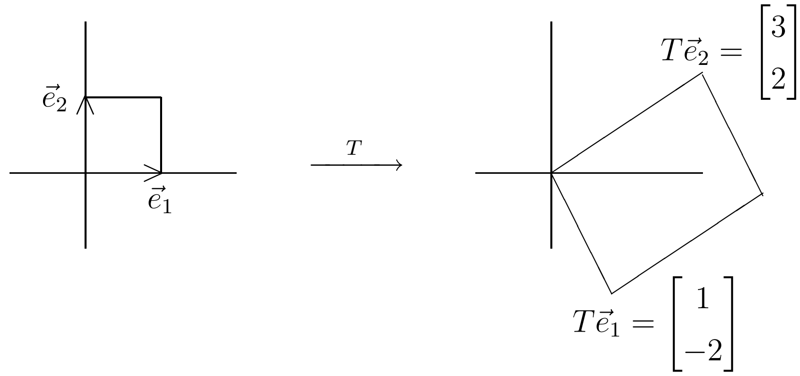
$$A \cdot \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ -2 & 1 \end{bmatrix}.$$

Hence:

$$A = \begin{bmatrix} 6 & 7 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & 7 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 8 & -\frac{17}{2} \\ 4 & -\frac{11}{2} \end{bmatrix}.$$

4. 4 points Sketch the image of the unit square under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \vec{x}$$



5. 6 points Find the matrices of the following linear transformations:

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, a clockwise rotation of 30°, followed by a dilation by a factor of 5.

$$A = 5 \cdot \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}}{2} & \frac{5}{2} \\ -\frac{5}{2} & \frac{5\sqrt{3}}{2} \end{bmatrix}$$

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, the reflection in the x–z-plane, followed by a dilation by a factor of 2.

$$A = 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

6. 5 points Find the projection of the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ onto the line L in the direction of the vector $\vec{w} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Unit vector in the direction of w:

$$\vec{u} = \frac{1}{||\vec{w}||} \cdot \vec{w} = \frac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}.$$

Dot product of $\vec{u}$ and $\vec{v}$:

$$\vec{u} \bullet \vec{v} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{8}{5} - \frac{3}{5} = 1.$$

Hence:

$$\text{proj}_L(\vec{v}) = (\vec{u} \bullet \vec{v}) \cdot \vec{u} = 1 \cdot \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$